KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 15 2018 08.00-13.00.

Answers and brief solutions.

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1. (a) i. The arbitrage bounds for the interest rate r are

$$0.5 \leq e^r \leq 1.5.$$

or if you use a simple rate

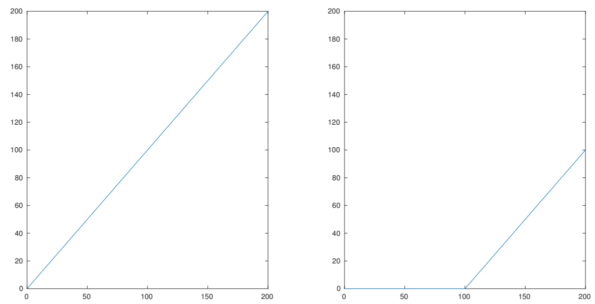
$$0.5 \leq (1 + r_s) \leq 1.5.$$

ii. Both the price of stock and the price of the option have to satisfy the risk-neutral valuation principle. This gives us the following set of equations

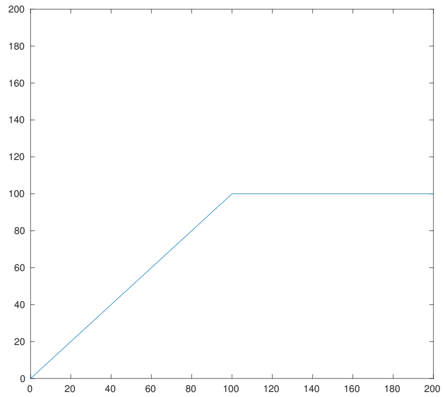
$$\begin{cases} 100 &= e^{-r}[q \cdot 150 + (1-q) \cdot 50], \\ 22 &= e^{-r}[q \cdot 42 + (1-q) \cdot 0]. \end{cases}$$

Solving these equations we find that r=4.8790% (and q=0.55), or r_s=5% if quoted as a simple rate.

(b) i. The payoff from the stock and the call option looks as follows



Since the stock has been bought and the call has been sold to create the covered call, the payoff from the covered call is given by:



In the figures above $K = 100$ has been used, but hopefully it should be clear what the payoff looks like for a general K.

ii. The portfolio is worth

$$100 - 2 = 98$$

to begin with, and at maturity it is worth

$$100 - 0 = 100.$$

It has therefore increased \$2 in value although the stock price has not moved.

2. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0942 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9139 \end{aligned}$$

and the tree for the futures price is therefore

		59.8609
	54.7087	
50.0000		50.0000
	45.6966	
		41.7635

Now the option price tree can be computed using

$$q = \frac{1-d}{u-d} \approx 0.4775,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0126}$$

and the result is

		0.0000
	1.0320	
3.7392		2.0000
	6.3034	
		10.2365

In each node the value is obtained as

$$\max\{52 - F_t, \frac{1}{1.0126}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where F_t is the current futures price, and P^u and P^d is the price of the option if the futures price goes up and down, respectively. Early exercise will be optimal in the node with option price 6.3034. The price of the option is thus 3.7392.

(b) If we denote the option price by Π we have that

$$\Delta = \frac{\partial \Pi}{\partial f} \approx \frac{\Delta \Pi}{\Delta f}.$$

This gives us

$$\Delta = \frac{1.0320 - 6.3034}{54.7087 - 45.6966} \approx -0.5849.$$

3. (a) i. We have that the futures price (which is equal to the forward price, since interest rates are deterministic) is given by

$$F(0,T) = \frac{\Pi_0(S_T)}{p(0,T)} = e^{rT} E^Q \left[\frac{S_T}{B_T} \right].$$

This futures price will make the initial value of the futures contract zero. To compute the expectation note that we have the formula

$$S_0 = E^Q \left[\frac{S_T}{B_T} + \sum_{t_i \le T} \frac{\Delta D_{t_i}}{B_{t_i}} \right]$$

so

$$S_0 = E^Q \left[\frac{S_{9/12}}{B_{9/12}} + \frac{\Delta D_{6/12}}{B_{6/12}} \right]. \quad (1)$$

If we use that $\Delta D_{6/12} = 3$ we obtain

$$S_0 = E^Q \left[\frac{S_{9/12}}{B_{9/12}} \right] + \frac{3}{B_{6/12-}} = E^Q \left[\frac{S_{9/12}}{B_{9/12}} \right] + e^{-0.05*0.5} 3$$

or

$$E^Q \left[\frac{S_{9/12}}{B_{9/12}} \right] = S_0 - e^{-0.05*0.5} 3$$

The futures price is therefore

$$F(0, 0.75) = e^{0.05 \cdot 0.75}(100 - e^{-0.05*0.5} 3) \approx 100.7835.$$

ii. Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)} F_t$) with parameters

$$F_0 = 100.7835, \quad K = 105, \quad \sigma = 0.20, \quad r = 0.05, \quad T = 0.25.$$

The price of the call option on the futures price is therefore

$$c_{fut}(0) = 2.3054.$$

The put-call parity for futures options reads

$$p_{fut}(0) = e^{-rT} K - e^{-rT} F_0 + c_{fut}(0).$$

This can be obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-rT} F_0$ everywhere. Using put-call parity we obtain

$$p_{fut}(0) = 6.4695.$$

(b) If we denote the price of a derivative written on the underlying stock by Π we have by definition that

$$\Delta = \frac{\partial \Pi}{\partial s}.$$

If we denote by $C(t, S_t)$ the price at time t of a European call option with strike price K and expiry date T written on the stock with price S_t at time t, and by $P(t, S_t)$ the price at time t of a European put option with the same strike price and expiry date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t, S_t) = Ke^{-r(T-t)} + C(t, S_t) - S_t$$

The price the synthetic long stock is therefore

$$\Pi = C(t, S_t) - P(t, S_t) = S_t - Ke^{-r(T-t)}$$

In the case of the synthetic long stock we therefore get

$$\Delta = \frac{\partial \Pi}{\partial s} = \frac{\partial s - Ke^{-r(T-t)}}{\partial s} = 1.$$

4. (a) i. Zero coupon bond prices satisfy

$$p^K(0, T_i) = e^{-r(0,T_i)T_i}K.$$

Here we have $T_1 = 0.5$, $K = 100$, and $r(0, 0.5) = 2.0\%$ and this yields the bond price

$$p^{100}(0, 0.5) = e^{-0.02 \cdot 0.5}100 = 99.0050.$$

ii. Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + Kp(t, T_n),$$

where $p(t, T) = p^1(t, T)$. For the two year coupon bond the coupon is $c^2 = 0.04 \cdot 0.5 \cdot 100 = 2$ and the formula reads

$$p^2_{fixed}(0) = 2p(0, 0.5) + 2p(0, 1) + 2p(0, 1.5) + (2 + 100)p(0, 2).$$

Using that $p(0, T_i) = e^{-r(0,T_i)T_i}$ this results in the coupon bond price

$$p^2_{fixed}(0) = 2e^{-r(0,0.5)0.5} + 2e^{-r(0,1)1} + 2e^{-r(0,1.5)1.5} + (2 + 100)e^{-r(0,2)2} = 99.0494.$$

iii. For the three year coupon bond the coupon is $c^3 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$p^3_{fixed}(0) = 3p(0, 1) + 3p(0, 2) + (3 + 100)p(0, 3).$$

Again using that $p(0, T_i) = e^{-r(0,T_i)T_i}$ this results in the coupon bond price

$$p^3_{fixed}(0) = 3e^{-r(0,1)1} + 3e^{-r(0,2)2} + (3 + 100)e^{-r(0,3)3} = 99.8026.$$

(b) The swap rate is set so as to make the value of the fixed and the floating leg equal, i.e.

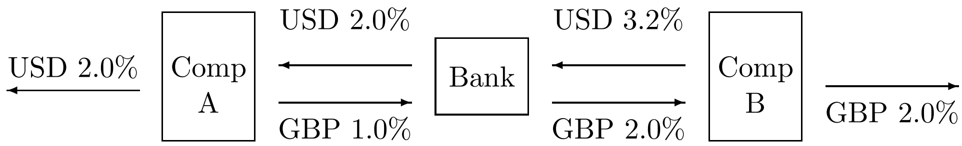
$$cp(0, 1) + cp(0, 2) + (c + K)p(0, 3) = K.$$

Using that $c = R_s \cdot 1 \cdot K$, we obtain

$$R_s = \frac{1 - p(0.3)}{p(0, 1) + p(0, 2) + p(0, 3)} = \frac{1 - 0.9139}{0.9753 + 0.9139 + 0.9139} \approx 0.0307$$

The swap rate is thus $R_s = 3.07\%$.

(c) Notice that the spread (the difference between the rate offered to Company B and the rate offered to Company A) in the US is 3.5-2.0 = 1.5%, whereas the spread in Great Britain is only 2.0-1.3 = 0.7%. The difference in spreads means that there is money to be made, ideally 1.5-0.7 = 0.8%. If the bank should get 0.2% this leaves 0.6% to the companies, and to be equally attractive to both the rates should be 0.3% lower for both companies. One way to set up the swap is the following:



This swap has the effect of transforming the USD interest rate of 2% per annum to a GBP interest rate of 1.0% per annum for Company A. So Company A is 0.3% per annum better than it would be if the swap had not been set up. From Company B's point of view the swap transforms the GBP interest rate of 2% per annum to a USD interest rate of 3.2% and ends up 0.3% per annum better than if the swap had not been set up. The bank makes a gain of 1.2% on its cash flows in USD, and a 1.0% loss on its cash flows in GBP. The net gain, ignoring exchange risk, is 0.2%. Thus the net gain to all parties is 0.3+0.3+0.2 = 0.8% as expected.

5. (a) The payoff X of the chooser option at time T_0 equals

$$X = \max\left\{C(T_0, S_{T_0}, K, T, r, \sigma), P(T_0, S_{T_0}, K, T, r, \sigma)\right\},$$

where $C(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. The notation $P(t, s, K, T, r, \sigma)$ is used for the price of the corresponding put option.

Using put-call-parity, $P(t, s, K, T, r, \sigma) = Ke^{-r(T-t)} + C(t, s, K, T, r, \sigma) - s$, this payoff can be written as

$$\begin{aligned} X &= \max\left\{C(T_0, S_{T_0}, K, T, r, \sigma), Ke^{-r(T-T_0)} + C(T_0, S_{T_0}, K, T, r, \sigma) - S_{T_0}\right\} \\ &= C(T_0, S_{T_0}, K, T, r, \sigma) + \max\left\{0, Ke^{-r(T-T_0)} - S_{T_0}\right\}. \end{aligned}$$

The price of the chooser option is therefore given by

$$\begin{aligned} \Pi(t; X) &= e^{-r(T_0-t)} E^Q\left[C(T_0, S_{T_0}, K, T, r, \sigma) + \max\{0, Ke^{-r(T-T_0)} - S_{T_0}\} \middle| \mathcal{F}_t\right] \\ &= e^{-r(T_0-t)} E^Q\left[C(T_0, S_{T_0}, K, T, r, \sigma) | \mathcal{F}_t\right] \\ &\quad + e^{-r(T_0-t)} E^Q\left[\max\left\{0, Ke^{-r(T-T_0)} - S_{T_0}\right\} \middle| \mathcal{F}_t\right]. \end{aligned}$$

Now, using that all price processes normalized by the risk free asset B are Q-martingales we find that $e^{-r(T_0-t)} E[C(T_0, S_{T_0}, K, T, r, \sigma) | \mathcal{F}_t] = C(t, S_t, K, T, r, \sigma)$. The second term in the price is easily identified as the price at time t of a put option, with exercise date T_0, and exercise price $Ke^{-r(T-T_0)}$. The price of the chooser option is thus given by

$$\Pi(t; X) = C(t, S_t, K, T, r, \sigma) + P(t, S_t, Ke^{-r(T-T_0)}, T_0, r, \sigma).$$

Both prices in the above formula can be explicitly computed using Black-Scholes formula, and put-call-parity.

(b) The price today ($t = 0$) of the T-claim X

$$X = \phi(S_T) = \left[\ln\left(\frac{S_T}{S_0}\right)\right]^2$$

is given by

$$\Pi_0 = e^{-rT} E^Q \left[\left[\ln\left(\frac{S_T}{S_0}\right)\right]^2\right].$$

Since $S_T = S_0 e^Z$ where $Z \in N\left((r - \sigma^2/2)T, \sigma^2 T\right)$ this can be written as

$$\Pi_t = e^{-rT} E^Q \left[Z^2\right].$$

This in turn can be computed as

$$\begin{aligned} \Pi_t &= e^{-rT}\left(V(Z) + \{E[Z]\}^2\right) \\ &= e^{-rT}\left(\sigma^2 T + (r - \sigma^2/2)^2 T^2\right). \end{aligned}$$