KTH Mathematics

Examination in SF2701 Financial Mathematics, May 23, 2008, 08:00–13:00.

Examiner: Filip Lindskog, tel. 790 7217, e-mail: lindskog@math.kth.se

Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Interest rates are continuously compounded.

GOOD LUCK!

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Problem 1

Maturity (years)	0.5	1	1.5	2
Annual coupon ($)	0	0	8	12
Bond price ($)	94.9	90.0	96.0	101.6
Face value ($)	100	100	100	100

Table 1: Specification of four different bonds.

Consider the bonds specified in Table 1. Half of the annual coupon is paid every six months, until and including the time of maturity, and the first coupon payment is in six months.

What is the price of a bond with an annual coupon of $10 maturing in two years with a face value of $200? (10 p)

Problem 2

Consider a bond portfolio consisting of one bond each of the bonds specified in Table 1. Black's formula is used to compute the price p of a European call option with maturity in just after 12 months and strike price $\$K$. Black's formula reads:

$$p = \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(G_0^{(t)}[X] e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}x}) e^{-\frac{1}{2}x^2} dx.$$

The yield at t years from now is, under the forward distribution, assumed to be normally distributed with variance $10^{-4}t$.

Determine the function f and the values of t, Z_t, $G_0^{(t)}[X]$ and σ that should go into Black's formula. (10 p)

Problem 3

The current spot price of a share which makes its first dividend payment of $2 in 9 months and a second dividend payment of $1 in 14 months is $20. The current forward price of one kilo of garlic to be delivered in one year is $2. The risk-free rate of interest is 5% per year.

What is the current forward price in kilos of garlic for delivery of the share in one year? (10 p)

Problem 4

A producer has negotiated a contract to sell 300′000 pounds of orange juice in April 2009 at the spot price at that time. To hedge the income the producer wants to take a position in orange juice futures contracts so that the variance of the effective price at which the producer sells (the hedged position) is minimized. Each futures contract is for delivery of 15′000 pounds of orange juice in May 2009. The current spot- and futures prices are 80 and 70 cents per pound, respectively.
The correlation coefficient between the spot- and futures price percentage returns (the April 2009 spot(futures) price divided by the current spot(futures) price) is assumed to be 0.7 and the standard deviations of the spot- and futures price percentage returns are assumed to be 0.6 and 0.4, respectively.
What futures position (long/short and size) should the producer take? (10 p)

Hint: The regression coefficient of Y onto X is $\mathrm{Cov}(X, Y)/\mathrm{Var}(X)$.

Problem 5

0	1	2	3	4
1300.00	1354.32	1410.92	1469.87	1531.30
	1225.44	1276.65	1330.00	1385.57
		1155.16	1203.43	1253.72
			1088.91	1134.41
				1026.46
0	1	2	3	4
37.6966	65.5022	110.916	169.874	231.295
	10.4586	21.0748	42.4668	85.5730
		0	0	0
			0	0
				0

Table 2: Binomial trees of spot- and derivative prices in \$. Each numerical value is rounded off to six digits. The time step is three months.

The upper tree shows spot prices of some asset from now until maturity of a derivative contract on this asset. The lower tree shows the corresponding derivative prices under the assumption of a constant risk-free interest rate. The binomial trees are contructed as in the lecture notes. Specify the derivate contract and the interest rate per year. (10 p)

Problem 6

To price interest rate securities a bank uses its internal discrete time short rate model where time is measured in years. The interest rate r_1 prevailing from now until one year from now is 6% and the interest rate r_2 for the following year is assumed to be either 4% or 8%.
Determine the price of a European call option with maturity in one year and strike price \$9′400 on a zero coupon bond that matures in two years with face value \$10′000. Today, the two-year zero coupon bond interest rate is 6.25% per year. (10 p)

Problem 1

Write r_t for the t-year zero rate. We have

$$\begin{aligned}
r_{0.5} &= -2\ln\left(\frac{94.9}{100}\right) = 0.1046930 \\
r_1 &= -\ln\left(\frac{90}{100}\right) = 0.1053605 \\
r_{1.5} &= -\frac{2}{3}\ln\left(\frac{96 - 4e^{-r_{0.5}0.5} - 4e^{-r_1}}{104}\right) = 0.1068093 \\
r_2 &= -\frac{1}{2}\ln\left(\frac{101.6 - 6e^{-r_{0.5}0.5} - 6e^{-r_1} - 6e^{-r_{1.5}1.5}}{106}\right) = 0.1080802.
\end{aligned}$$

Hence, the price of the new bond is

$$5e^{-r_{0.5}0.5} + 5e^{-r_1} + 5e^{-r_{1.5}1.5} + 205e^{-r_2 2} = 178.6541.$$

Problem 2

Write P_t for the value of the bond portfolio at time t (years). Forward price for delivery in just after one year:

$$G_0^{(1)}[P_1] = Z_1^{-1}P_0^{(1)}[P_1] = e^{r_1}(110e^{-r_{1.5}1.5} + 106e^{-r_2 2}) = 199.0111.$$

Forward yield y_{F}:

$$G_0^{(1)}[P_1] = 110e^{-y_{\mathrm{F}}0.5} + 106e^{-y_{\mathrm{F}}} \quad \Rightarrow y_{\mathrm{F}} = 0.1104.$$

Forward duration D_{F}:

$$D_F G_0^{(1)}[P_1] = 110 \cdot 0.5 \cdot e^{-y_{\mathrm{F}}0.5} + 106e^{-y_{\mathrm{F}}} \quad \Rightarrow D_{\mathrm{F}} = 0.738485.$$

So Black's formula should be used with $f(x) = \max(x - K, 0)$ and the values: $t = 1$, $Z_t = 0.9$, $G_0^{(t)}[X] = \$199.0111$ and $\sigma = 0.00738485$.

Problem 3

The forward price is

$$G_0^N[X] = \frac{P_0^{(1)}[X]}{P_0^{(1)}[N]} \text{ kilos of garlic.}$$

We have $P_0^{(1)}[N] = Z_1 G_0^{(1)}[N] = \$e^{-0.05}2$ and $P_0^{(1)}[X] = \$20 - 2e^{-0.05\cdot 0.75}$. Hence, the forward price is

$$G_0^N[X] = e^{0.05}(10 - e^{-0.05\cdot 0.75}) \approx 9.5 \text{ kilos of garlic.}$$

Problem 4

Let today be time 0, let time t be the contracted date in April. The hedge should be a short position in the orange juice futures contracts, closed out at time t. We have to determine how many futures contracts should be shorted. In order to do this we may determine the regression coefficient β of the percentage return of the orange juice spot price onto the percentage return of the orange juice futures price:

$$S_t/S_0 = \beta F_t/F_0 + e. \tag{1}$$

From standard theory of linear regression we know that

$$\beta = \frac{\mathrm{Cov}(S_t/S_0, F_t/F_0)}{\mathrm{Var}(F_t/F_0)} = \mathrm{Corr}(S_t/S_0, F_t/F_0)\sqrt{\frac{\mathrm{Var}(S_t/S_0)}{\mathrm{Var}(F_t/F_0)}} = 0.7\frac{0.6}{0.4} = \frac{21}{20}.$$

This choice of β minimizes the variance $\mathrm{Var}(e)$ of the residual term e. Moreover, it is easily checked that this choice of β implies that $\mathrm{Cov}(F_t/F_0, e) = 0$. The effective price per pound at time t for the orange juice is the spot price at time t plus the gain (or loss) per pound from the short futures position: $S_t + kn(F_0 - F_t)$, where $n = 15'000$ and k is the number of shorted futures contracts per pound of orange juice sold. Using (1) we can write this price as

$$\begin{aligned}
S_t + kn(F_0 - F_t) &= \beta\frac{S_0}{F_0}F_t + S_0 e + kn(F_0 - F_t) \\
&= \left(\beta\frac{S_0}{F_0} - kn\right)F_t + knF_0 + S_0 e \\
&= \beta S_0 + S_0 e
\end{aligned}$$

when choosing

$$k = \frac{\beta S_0}{nF_0} = \frac{21}{20}\frac{8}{7}\frac{1}{n} = \frac{6}{5n}.$$

Since the producer will sell $300'000$ pounds of grape fruit, the total number of futures contracts to short is $300'000k = 24$.

Problem 5

American call option with strike price \$1300 and maturity in one year. The interest rate is 3% per year.

Problem 6

We have

$$e^{-0.125} = Z_2 = \widehat{\mathrm{E}}[e^{-r_1-r_2}] = e^{-0.06}(qe^{-0.04} + (1-q)e^{-0.08})$$

so $q = (e^{-0.025} - e^{-0.04})/(1 - e^{-0.04}) \approx 0.37$. We have $Z_{1,2} = \widehat{\mathrm{E}}_1[e^{-r_2}] = e^{-r_2}$ and

$$\begin{aligned}
P_0^{(1)}[\$10000\max(Z_{1,2} - 0.94, 0)] &= 10000F_0^{(1)}[\$\max(e^{-r_2} - 0.94, 0)e^{-0.06}] \\
&= \$10000\widehat{\mathrm{E}}[\max(e^{-r_2} - 0.94, 0)e^{-0.06}] \\
&= \$q10000(e^{-0.04} - 0.94)e^{-0.06} \approx \$72.50414.
\end{aligned}$$