KTH Teknikvetenskap

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Exam, 26/5-2009, 8.00–13.00.
SF2701 Financial mathematics – Basic Course.
Aid: pocket calculator

Number the pages, and write your name on each sheet. Interest rates are always with continuous compounding. The exam will be marked as an entirety, so no "credits" are given for the individual problems.

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1. The one year interest rate is today 0.04. The one year interest rate one year from now (i.e., the interest we get on a deposit we make one year from now and withdraw two years from now) is modeled as

$$r_{1,2} = 0.045 + 0.01z \quad z \in N(0,1)$$

in the 2-year forward probability distribution. Determine today's two-year zero rate (with five significant digits!) Motivate carefully!

2. A bond pays dividends \$30 every six months and matures in three years with face value \$1'000 (which means that \$1'030 is payed out, including the dividend.)

Determine the *forward price* of the bond to be delivered in one year just after the dividend payment, and also the *forward yield* and the *forward duration* at that time. The current zero rates (per year) are as follows:

months	6	12	18	24	30	36
rate %	4.00	4.50	5.00	5.25	5.40	5.50

3. A share of a stock costs today \$40. In 5 weeks it will give a dividend of \$3. The volatility of the share is assumed to be 0.25 in one year (which we can say is 50 weeks.) The interest rate is 7% per year.

Construct a biomial tree of spot prices of this asset with time step two weeks ending in eight weeks from now. Use a precision of four decimal places. (If you use any other kind of tree than we have used in this course, you must specify the "up" and "down" probabilities!)

4. Here is a Ho-Lee binomial tree of interest rates (% per period; time goes from bottom to top; "now" is period 0)

4.6	4.2	3.8	3.4
4.5	4.1	3.7	
4.3	3.9		
4.0			

Determine the *forward* price of an interest security which in period 3 (= time of delivery; the top row of the Ho-Lee tree) is worth

100	95	90	85

(The value 100 pertains to the situation that the interest rate from period 3 to period 4 is 4.6% and so on.)

Continued . . .

5. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner \$100 if $S(1) > \$50$ and nothing if $S(1) \leq \$50$. The current spot price is $S(0) = \$45$; the share's volatility is assumed to be 30% for one year and the risk-free rate of interest is 6% a year. (If your pocket calculator doesn't have the cumulative normal distribution function $\Phi(\cdot)$, your answer may be expressed in terms of $\Phi(\cdot)$.)

6. Let X be the value of a share of a stock one year from now, and let e^R be the balance one year from now on a "money market account" (MMA) if we deposit 1 today, i.e., $R = \sum_1^{364} r_i$ where r_i is the floating daily interest from day $i-1$ to day i. Under the "futures distribution" (the "traditional riskfree probability distribution" in Hull's terminology) the specification of X and R are as follows:

$$X = Ae^{\sigma z}, \qquad R = a + \eta w$$

where A and a are unspecified constants[1]; σ and η are specified constatns ("volatilities";) $z \in N(0,1)$, $w \in N(0,1)$, and z and w are independent.

Black's pricing formula for a call option can be written as

$$B(Z_t, G_0, \sigma, t, K) = Z_t \, \mathrm{E}\left[\left(G_0 e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,u} - K\right)^+\right], \quad u \in N(0,1)$$

Determine the price of a call option on X with strike price Ke^R (the strike price is hence stochastic; it is the balance of a MMA with deposit K today) expressed as $B(\cdot,\cdot,\cdot,\cdot,\cdot)$, i.e., give the correct expressions for the parameters that correspond to the five parameters in Black's formula. The share gives a dividend ρS after six months (S = spot price at that time,) the spot price today is S_0, the six months and twelve months interest rates are r_6 and r_{12} respectively. The answer may contain the parameters S_0, K, r_6, r_{12}, ρ, σ and η.

Answers

1. 4.2525%
2. $G_0 = 998.31$, $y_F = 0.06$, $D_F = 1.914$ yr.
3. Time from bottom and up:

45.499	41.169	37.251	33.706	30.499
43.213	39.100	35.380	32.013	
44.037	40.132	36.598		
41.967	38.257			
40.000				

4. 92.485
5. $\$e^{-0.06}100(1 - \Phi(0.301202)) = \$35.94$.
6. $B(1, (1-\rho)S_0, \sqrt{\sigma^2 + \eta^2}, 1, K)$.

[1] This means that you can not have them in the answer.