KTH Mathematics

Examination in SF2701 Financial Mathematics, May 29, 2010, 09:00–14:00.

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*Allowed technical aids*: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Interest rates are continuously compounded. Binomial trees should be constructed as in the lecture notes. Figure 2 shows a plot of the $N(0,1)$ distribution function.

GOOD LUCK!

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### Problem 1

You are thinking about taking a long butterfly position on the value of a stock index one year from now. More precisely, you are thinking about buying a European derivative on the value of the index one year from now with the payoff function shown in Figure 1. The current prices of European call options on the index one year from now are shown in Table 1. The current value of the index is 97. You believe that the value of the index one year from now can be modeled as the random variable

$$100e^{-0.02+0.2Z}, \quad \text{where } Z \sim N(0,1).$$

You decide to take the position if, according to your beliefs, the probability that the long butterfly position will be profitable is above 50%. Should you take the long butterfly position? (10 p)

| Strike price | 90 | 95 | 100 | 105 | 110 |
|---|---|---|---|---|---|
| Option price | 14.33 | 11.24 | 8.63 | 6.49 | 4.78 |

Table 1: Call option prices.

### Problem 2

Table 2 below shows an incomplete binomial tree of forward prices on the spot price of an asset one year from now. The forward price tree has a time step of three months. Determine the probability distribution of the forward price six months from now (the possible values it can take and the probabilities) according to the binomial tree in Table 2. (10 p)

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| ? | ? | ? | ? | ? |
| | ? | ? | 54.43721 | ? |
| | | ? | 44.56942 | 49.01156 |
| | | | ? | ? |
| | | | | ? |

Table 2: Incomplete forward price tree.

### Problem 3

Determine the price of an American put opion on a share whose current spot price is 100 with maturity in one year and with strike price 104. The share's volatility is 20% in one year, the interest rate is 5% a year. The share does not pay dividends. Use a binomial tree with a time step of three months.
You may use the following information: $\tanh(0.05) = 0.04995837$, $\tanh(0.1) = 0.099668$, $\tanh(0.2) = 0.1973753$. (10 p)

### Problem 4

Determine the price of a European derivative on the spot price one year from now of one share of a dividend paying stock. The payoff function $f(x)$ of the derivative is 1 if $x \in [100, 105]$ and 0 otherwise. The current share price is 100 and the interest rate is 5% a year. The share pays a dividend of 4 in three months. Use Black's model and assume that the share's volatility is 20% in one year. (10 p)

### Problem 5

A company will buy orange juice in April 2011 at the spot price at that time. The number of pounds of orange juice it needs to buy is unknown today but can be regarded as a random variable with expected value $150'000$ and standard deviation $10'000$. To hedge the uncertain cost the company wants to take a position in orange juice futures contracts so that the variance of the effective price (including the gain/loss from the futures position) for buying the orange juice is minimized. Each futures contract is for delivery of $15'000$ pounds of orange juice in May 2011. The current spot- and futures prices are 90 and 70 cents per pound, respectively.
The correlation coefficient between the April 2011 spot- and futures prices is 0.9. The expected value and standard deviation of the April 2011 spot price are 80 and 20 cents, respectively. The expected value and standard deviation of the April 2011 futures price are 80 and 15 cents, respectively. The quantity that that company needs to buy in April 2011 is assumed to be independent of the spot- and futures prices at that time.
What futures position (long/short and size) should the company take? (5 p)

What is the standard deviation of the effective price for the orange juice that the company will buy? (5 p)

### Problem 6

Consider Ho-Lee's short rate model:

$$r_1 = -\log Z_1, \quad r_k = \log\left(\frac{Z_{k-1}}{Z_k}\right) + \frac{\sigma^2}{2}(k-1)^2 + \sigma(W_1 + \cdots + W_{k-1}), \quad k \geq 2,$$

where the $Z_j$s are the current zero-coupon bond prices, and the $W_j$s are independent and $N(0,1)$-distributed (under the futures distribution) and the outcome of $W_j$ occurs at time $j$. Determine the price now at time 0, in terms of the $Z_j$s and $\sigma$, of an interest rate security that pays $e^{r_2}$ at time 3. (10 p)

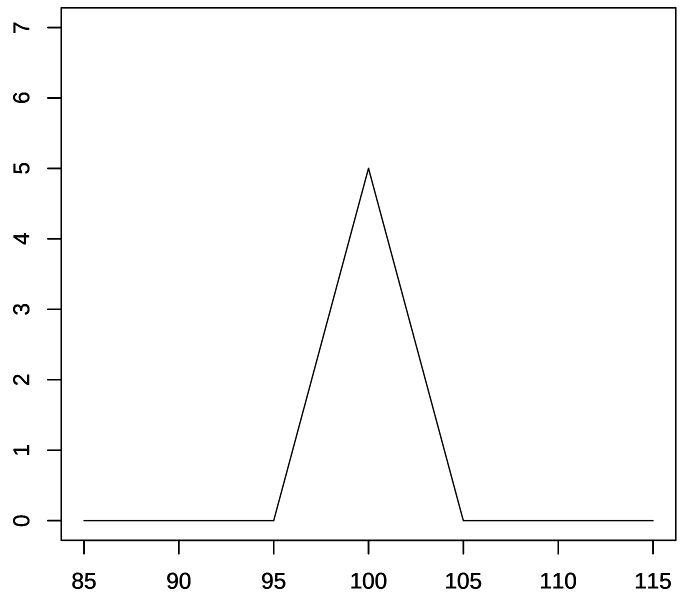


Figure 1: The butterfly payoff function.

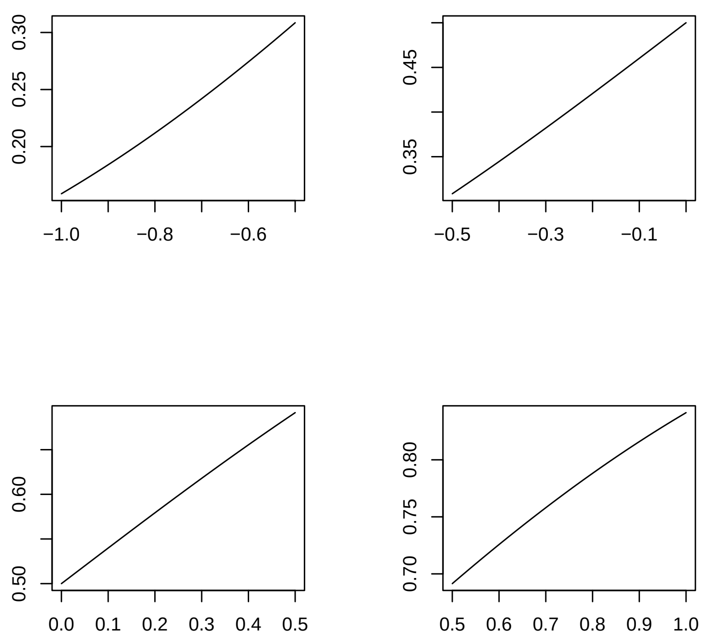


Figure 2: Plot of the standard normal distribution function $\Phi$.

## Problem 1

The butterfly position corresponds to a long position of one call with skrike 95, a short position of two calls with strike 100 and a long position of one call with strike 105. The price of the long butterfly position is thus $11.24 - 2 \cdot 8.63 + 6.49 = 0.47$. The butterfly position is profitable if

$$\max(I - 95, 0) - 2\max(I - 100, 0) + \max(I - 105, 0) - 0.47 \geq 0,$$

or equivalently if $I \in (95.47, 104.53)$. The probability of this event, according to your view, is

$$\begin{aligned}
&\Pr(100e^{-0.02+0.2Z} \leq 104.53) - \Pr(100e^{-0.02+0.2Z} \leq 95.47) \\
&\quad = \Phi\Big(5\big(\log(1.0453) + 0.02\big)\Big) - \Phi\Big(5\big(\log(0.9547) + 0.02\big)\Big) \\
&\quad = \Phi(0.3215196) - \Phi(-0.1317906) \\
&\quad \approx 0.63 - 0.45 = 0.18.
\end{aligned}$$

You should not take this position (according to your criterion).

## Problem 2

Table 3 shows the complete forward price tree. According to this tree the probability distribution of the forward price six months from now is 60.46348 with probability 0.25, 49.50331 with probability 0.5, and 40.52989 with probability 0.25.

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 50 | 54.9834 | 60.46348 | 66.48976 | 73.11666 |
| | 45.0166 | 49.50331 | 54.43721 | 59.86286 |
| | | 40.52989 | 44.56942 | 49.01156 |
| | | | 36.49035 | 40.12727 |
| | | | | 32.85343 |

Table 3: Complete forward price tree.

## Problem 3

The first tree in Table 4 is the tree of forward prices, where $G_0$ is the current forward price for delivery of the share in one year and $u = 1 + \tanh(0.1) = 1.099668$ and $d = 1 - \tanh(0.1) = 0.900332$. The second tree is that tree with numerical values inserted. The third tree is the corresponding tree of spot prices, where $\gamma = \exp\{r\Delta t\} = 1.012578$. The fourth tree is the same tree with numerical values inserted. The fifth tree is the tree of European put option prices (no necessary here). Hence, the European put option price is 7.083905. The sixth tree is the tree of American put option prices. Hence, the American put option price is 8.021098.

## Problem 4

The forward price is $G_0 = G_0^{(1)}[S_1] = 100.9743$. Assuming Black's model gives that, under the forward probability distribution,

$$S_1 = G_0 e^{-\sigma^2/2+\sigma Z}, \quad Z \sim N(0, 1),$$

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $G_0$ | $G_0u$ | $G_0u^2$ | $G_0u^3$ | $G_0u^4$ |
| | $G_0d$ | $G_0du$ | $G_0du^2$ | $G_0du^3$ |
| | | $G_0d^2$ | $G_0d^2u$ | $G_0d^2u^2$ |
| | | | $G_0d^3$ | $G_0d^3u$ |
| | | | | $G_0d^4$ |

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 105.1271 | 115.6049 | 127.1270 | 139.79752 | 153.7309 |
| | 94.6493 | 104.0828 | 114.45653 | 125.8642 |
| | | 85.2158 | 93.70908 | 103.0489 |
| | | | 76.72251 | 84.36929 |
| | | | | 69.07573 |

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $S_0$ | $S_0u\gamma$ | $S_0u^2\gamma^2$ | $S_0u^3\gamma^3$ | $S_0u^4\gamma^4$ |
| | $S_0d\gamma$ | $S_0du\gamma^2$ | $S_0du^2\gamma^3$ | $S_0du^3\gamma^4$ |
| | | $S_0d^2\gamma^2$ | $S_0d^2u\gamma^3$ | $S_0d^2u^2\gamma^4$ |
| | | | $S_0d^3\gamma^3$ | $S_0d^3u\gamma^4$ |
| | | | | $S_0d^4\gamma^4$ |

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 100 | 111.3500 | 123.9883 | 138.0609 | 153.7309 |
| | 91.16568 | 101.513 | 113.0347 | 125.8642 |
| | | 83.11181 | 92.54501 | 103.0489 |
| | | | 75.76945 | 84.36929 |
| | | | | 69.07573 |

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 7.083905 | 2.707059 | 0.2319043 | 0 | 0 |
| | 11.63896 | 5.250315 | 0.4696426 | 0 |
| | | 18.32041 | 10.16307 | 0.9511 |
| | | | 26.93864 | 19.63071 |
| | | | | 34.92427 |

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 8.021098 | 3.02207 | 0.2319094 | 0 | 0 |
| | 13.22191 | 5.8882564 | 0.4696426 | 0 |
| | | 20.88819 | 11.45499 | 0.9511 |
| | | | 28.23055 | 19.63071 |
| | | | | 34.92427 |

Table 4: Put option prices.

where $\sigma = 0.2$ by assumption. The derivative price is

$$\begin{aligned}
P_0^{(1)}[I\{S_1 \in [100, 105]\}] &= e^{-0.05}\,\mathrm{E}^{(1)}[I\{S_1 \in [100, 105]\}] \\
&= e^{-0.05}\,\mathrm{Pr}^{(1)}(G_0 e^{-\sigma^2/2+\sigma Z} \in [100, 105]) \\
&= e^{-0.05}\Phi\big((\log(105/G_0) + 0.02)/0.2\big) \\
&\quad - e^{-0.05}\Phi\big((\log(100/G_0) + 0.02)/0.2\big) \\
&= e^{-0.05}\Phi(0.2954716) - e^{-0.05}\Phi(0.05152078) \\
&\approx 0.0910.
\end{aligned}$$

## Problem 5

Let $X$ be the unknown quantity, let $S$ be the spot price in April 2011, let $F$ be the futures price in April 2011, let $F_0$ be the current futures price, let $k = 15'000$, and let $n$ be the position in the futures contract. The effective price is

$$S^e = XS - nk(F - F_0).$$

Write $\widetilde{S} = XS$, $\widetilde{F} = kF$, and $\widetilde{F}_0 = kF_0$. Then $S^e = \widetilde{S} - n(\widetilde{F} - \widetilde{F}_0)$ and we know that the choice of $n$ that minimizes the variance of $S^e$ is

$$n = \frac{\operatorname{Cov}(\widetilde{S}, \widetilde{F})}{\operatorname{Var}(\widetilde{F})}.$$

Since $\operatorname{Cov}(X, S) = 0$ we have

$$\operatorname{Cov}(\widetilde{S}, \widetilde{F}) = \operatorname{E}[X] \operatorname{Cov}(S, \widetilde{F}) = k \operatorname{E}[X] \operatorname{Cov}(S, F) = k \operatorname{E}[X] \operatorname{sd}(S) \operatorname{sd}(F) \operatorname{corr}(S, F).$$

This gives

$$n = \frac{\operatorname{E}[X]}{k} \frac{\operatorname{sd}(S)}{\operatorname{sd}(F)} \operatorname{corr}(S, F) = 10 \frac{20}{15} \frac{9}{10} = 12.$$

Moreover, the variance of the effective price is

$$\begin{aligned}
\operatorname{var}(S^e) &= \operatorname{Var}(\widetilde{S}) - 2n \operatorname{Cov}(\widetilde{S}, \widetilde{F})) + n^2 \operatorname{Var}(\widetilde{F}) \\
&= \operatorname{Var}(\widetilde{S}) - \frac{\operatorname{Cov}(\widetilde{S}, \widetilde{F})^2}{\operatorname{Var}(\widetilde{F})} \\
&= \operatorname{Var}(\widetilde{S}) - n \operatorname{Cov}(\widetilde{S}, \widetilde{F}) \\
&= \operatorname{Var}(\widetilde{S}) - nk \operatorname{E}[X] \operatorname{sd}(S) \operatorname{sd}(F) \operatorname{corr}(S, F).
\end{aligned}$$

We have

$$\begin{aligned}
\operatorname{Var}(\widetilde{S}) = \operatorname{Var}(XS) &= \operatorname{E}[X^2] \operatorname{E}[S^2] - \operatorname{E}[X]^2 \operatorname{E}[S]^2 \\
&= \operatorname{Var}(X) \operatorname{Var}(S) + \operatorname{Var}(X) \operatorname{E}[S]^2 + \operatorname{Var}(S) \operatorname{E}[X]^2 \\
&= 10^8 (0.2)^2 + 10^8 (0.8)^2 + (0.2)^2 (1.5 \cdot 10^5)^2 \\
&= 9.68 \cdot 10^8.
\end{aligned}$$

and therefore

$$\operatorname{Var}(S^e) = (9.68 - 7.29) 10^8 = 2.39 \cdot 10^8 \quad \text{and} \quad \operatorname{sd}(S^e) = 15,459.62.$$

## Problem 6

The price of the interest rate security is

$$\begin{aligned}
P_0^{(3)}[e^{r_2}] &= F_0^{(3)}[e^{r_2} e^{-r_1 - r_2 - r_3}] \\
&= F_0^{(3)}[e^{-r_1 - r_3}] \\
&= \widehat{\operatorname{E}}[e^{-r_1 - r_3}] \\
&= e^{\log(Z_1) + \log(Z_3/Z_2)} e^{-2\sigma^2} \widehat{\operatorname{E}}[e^{-\sigma W_1}] \widehat{\operatorname{E}}[e^{-\sigma W_2}] \\
&= Z_1 \frac{Z_3}{Z_2} e^{-2\sigma^2} e^{\sigma^2/2} e^{\sigma^2/2} \\
&= Z_1 \frac{Z_3}{Z_2} e^{-\sigma^2}.
\end{aligned}$$