KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 20 2014 14.00-19.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) A stock price is currently \$50. It is known that at the end of three months it will be either \$54 or \$47. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a three-month European put option with a strike of \$51? ....(5p)

(b) Consider the standard Black-Scholes setting. The stock price is currently \$78 and the risk free rate is 3% per annum with continuous compounding. A six-month put option on the stock with a strike price of \$80 is currently worth \$6.5. What is the value of a six-month call on the stock with a strike price of \$80 if both the put and the call are European options? ....(5p)

2. (a) Compute the price of an American call option written on a stock which pays a continuous dividend yield of 5%. The current stock price is \$100 and the volatility of the stock price is 25%. The maturity of the option is in six months and the strike price is \$100. The risk free interest rate with continuous compounding is 3% per annum. You should use a two period binomial model to price the option. . . . . (8p)

(b) If the underlying stock were not to pay a dividend before the maturity of the option, would this increase or decrease the value of the call option? Please motivate! . . . . (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding.

(a) i. Consider a nine-month European call option written on a stock index paying a continuous dividend yield of 4%. The value of the stock index is currently 100, the strike price of the option is 99, the risk-free interest rate is 3% per annum, and the volatility of the index is 20%. Compute the price of the European call option. . . . . (3p)

ii. Suppose instead that the option in exercise 3(a)i is American. Can you then price it in a continuous time framework? If you can, compute the price of the option, if you can not, state conditions under which you would be able to price it (in a continuous time framework!). . . . . (2p)

(b) Consider a six-month European put option written on a futures contract. The futures contract is written on one barrel of crude oil and has delivery date in six months. The strike price of the futures option is \$103 and the current spot price of crude oil is \$102 per barrel, the six-month futures price of one barrel of crude oil is \$100, the risk-free interest rate is 5% per annum, and the volatility of the spot price is 15%, whereas the volatility of the futures price is 16.5%. Compute the price of the European put option. . . . . (5p)

4. (a) Suppose that the following bonds are currently trading in the market:

- A zero coupon bond with face value 100 and maturity one year trades at 97.04
- A coupon bond with face value 100, maturity in two years, and a coupon of 3% per annum paid annually trades at 99.9063
- A coupon bond with face value 100, maturity in three years, and a coupon of 2% per annum paid annually trades at 95.104

i. Compute the current term structure, i.e. the zero rates for one year, two years, and three years. Quote the rates per annum with continuous compounding. . . . . (3p)

ii. Compute the one-year forward rates for years two and three. Quote the rates per annum with continuous compounding. . . . . (3p)

(b) Using the term structure in exercise 4a, value a forward rate agreement where you will pay 5% (compounded annually) for the third year on \$1 million. (4p)

5. Consider the standard Black-Scholes model.

(a) Determine today's arbitrage price of a so called *straddle*, which is a contingent T-claim defined by

$$X = \phi(S_T) = |S_T - K|.$$

A straddle is thus a contract you would buy if you believed that there would be large movements in the stock price, but you were not sure in which direction.

The answer is allowed to be expressed in terms of the cumulative distribution function of a normally distributed random variable with expectation zero and variance one and the parameters of the problem (K, μ (local mean rate of return of the stock under P), σ, r, T, S_0). .... (5p)

Hint: Draw the contract function!

(b) Determine today's arbitrage price of a contingent T-claim defined by

$$X = \phi(S_T) = S_T^2.$$

The answer is allowed to be expressed in terms of the parameters of the problem: μ (local mean rate of return of the stock under P), σ, r, T, S_0. .... (5p)

Hint: You may find some of the hints at the end of the exam useful to solve this exercise.

Good luck!

Hints:

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N$(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- If Y is lognormally distributed random variable, i.e. $Y = e^Z$ where $Z \in N(\mu, \sigma)$ (so Z has expectation μ and variance σ^2), then

$$E[Y] = e^{\mu+\sigma^2/2} \quad \text{and} \quad V(Y) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}.$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

  $$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

  $$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \leq x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$