KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 19 2015 8.00-13.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) A stock price is currently \$60. It is known that at the end of three months it will be either \$65 or \$55. The risk-free interest rate is 3% per annum with continuous compounding. What is the value of a three-month European call option with a strike of \$60? . . . . (5p)

(b) An option trading strategy referred to as a *top straddle* or *straddle write* involves selling a European call and put option on the same underlying stock, with the same strike price and expiration date. Suppose the strike price of the options is 100, and that the call trades at 9, and the put at 8.

Draw the net payoff function of the straddle described above, i.e the payoff function plus initial earnings. To avoid having to discount assume that the interest rate is zero. . . . . (5p)

2. (a) Compute the price of an American put option to buy US dollars (USD) for Japanese yen (JPY). The current exchange rate is 125 JPY/USD and the volatility of the exchange rate is 10%. The maturity of the option is in six months and the strike price is 130 JPY. The dollar interest rate with continuous compounding is 2% per annum, and the yen interest rate with continuous compounding is 4% per annum. You should use a two period binomial model to price the put option. .... (8p)

(b) Estimate the current value of the delta of the option using your tree. ... (2p)

3. In this exercise you should work in the Black-Scholes frame work, and all interest rates are quoted per annum with continuous compounding.

(a) i. State put-call parity in the standard Black-Scholes frame work. ..... (2p)

ii. Use the definition of delta and the put-call parity to compute delta for a European put option on a non-dividend paying stock (you may use what delta for the corresponding call option is without proving it). The exercise date of the option is T and the strike price is K. .... (2p)

(b) A stock is expected to pay a dividend of 5% of its value in three months time. The stock price is \$100, and the risk-free rate of interest is 2% per annum with continuous compounding. An investor has just taken a long position in a six-month forward contract on the stock.

i. What are the forward price and the initial value of the forward contract? .... (3p)

ii. Consider a six-month European call option written on the stock. The strike price of the option is 95, and the volatility of the stock is 20%. Compute the price of the European call option. .... (3p)

Hint: The volatility of the forward price is the same as the volatility of the stock price, and when interest rates are deterministic forward prices are equal to futures prices.

4. (a) Suppose the current term structure looks as follows:

T	$r(0,T)$
0.5	1.5%
1	2.0%
2	2.5%

Here $r(0,T)$ denotes the current zero rate for the time period T. Compute the prices of

i. A zero coupon bond with face value 100 and maturity six months,

ii. a zero coupon bond with face value 100 and maturity one year,

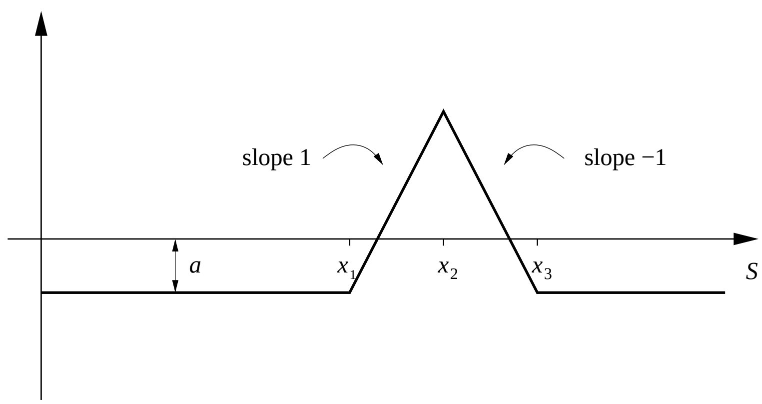
iii. a coupon bond with face value 100, maturity in two years, and a coupon of 3% per annum paid annually.

.... (3p)

(b) Given the term structure in (a), compute the swap rate of a two year fixed-for-floating swap, where six-month LIBOR is to be exchanged for the fixed swap rate R_s. Floating rate payments are to be made semi-annually, whereas the fixed payments are to be made once a year. The notational principal of the swap is one million dollars. The swap rate R_s should be quoted per annum using annual compounding. .... (3p)

(c) Using the term structure in exercise 4a, value a forward rate agreement where you will pay 3.5% (compounded annually) for the second year on \$1 million. .... (4p)

5. (a) Consider the standard Black-Scholes model. Suppose that you for some reason are fairly certain that the stock price will not move much until time T. In fact you are so certain of this you are willing to bet on it. Thus you would like to enter a *butterfly spread*, which is a T-contract with the payoff structure depicted in the figure below.



A natural choice for x_2 is today's stock price S_0 and given that you have no information about whether a rise in the stock price is more likely than a fall you would set $x_2 = (x_1 + x_3)/2$. How narrow you want the interval $[x_1, x_3]$ depends on how sure you are that the stock price will not move. For this exercise set $x_1 = 0.95S_0$ (and thus $x_3 = 1.05S_0$).

Now suppose that you do not have much money at the moment, and therefore would not want to pay anything entering the contract. Furthermore, suppose that today's stock price is 100, T is three months, the interest rate with continuous compounding is 2%, and the volatility of the stock 20%. Determine how much you must be willing to loose at most, i.e. the constant a, if you do not want to pay anything for the contract today. .... (5p)

Exercise (b) on next page.

(b) In his thesis from 1900 Louis Bachelier suggested the following model for the (discounted) stock price under the martingale measure Q

$$dS_t = S_0 \sigma dV_t.$$

Here S_0 is the initial value of the stock, σ is a constant, and V denotes a Q-Wiener process. This means that

$$S_T = S_0 + S_0 \sigma V_T \stackrel{d}{=} S_0 + S_0 \sigma Z.$$

Here the superscript d means that the equality holds in distribution, i.e. S_T has the same distribution as $S_0 + S_0\sigma Z$, where $Z \in N(0, T)$ (here the second parameter T is the variance).

Suppose that the interest rate is zero so that we do not have to worry about discounting. For $t = 0$ derive the following option pricing formula for a European call option on the stock with price process S, strike price K, and exercise date T

$$C_{Bach}(0) = (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma S_0 \sqrt{T}}\right) + \sigma S_0 \sqrt{T} \varphi\left(\frac{S_0 - K}{\sigma S_0 \sqrt{T}}\right). \tag{1}$$

Here Φ and φ denote the standard normal distribution and density function, respectively, i.e.

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \varphi(u) du.$$

. . . . (5p)

Hint: You may, without proving it, use the following result about normal distributions: If $X \in N(\mu, \sigma^2)$ then

$$E[X I_{\{l \le X \le h\}}] = \int_l^h x\varphi(x)dx =$$

$$= \mu\left[\Phi\left(\frac{h-\mu}{\sigma}\right) - \Phi\left(\frac{l-\mu}{\sigma}\right)\right] + \sigma\left[\varphi\left(\frac{l-\mu}{\sigma}\right) - \varphi\left(\frac{h-\mu}{\sigma}\right)\right]$$

where I_A denotes the indicator function of the set A.

Good luck!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \leq x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$