KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday May 30 2018 14.00-19.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) A stock price is currently \$50. It is known that at the end of three months it will be either \$54 or \$47. The risk-free interest rate is 5% per annum with continuous compounding. Compute the replicating portfolio for a three-month European put option with a strike of \$51. .... (5p)

(b) Consider the standard Black-Scholes setting. Now consider the portfolio corresponding to

- buying a call with strike price K and selling a put with strike price K.

Both options are European, written on the stock, and have the same expiry date T. A portfolio strategy like this is known as a *synthetic long stock*.

i. Draw the payoff function of the *synthetic long stock*. .... (2p)

ii. Determine the arbitrage price of the *synthetic long stock*, given that the current stock price is \$86, the strike price K is \$85, and the expiry date is in three months. Furthermore the volatility of the stock is 30%, and the risk free interest rate with continuous compounding is 2% per annum. (3p)

2. (a) Compute the price of an American call option to buy US dollar (USD) for Japanese yen (JPY). The current exchange rate is 110 JPY/USD and the volatility of the exchange rate is 10%. The maturity of the option is in six months and the strike price is 105 JPY. The dollar interest rate with continuous compounding is 4% per annum, and the yen interest rate with continuous compounding is 1% per annum. You should use a two period binomial model to price the call option. . . . . (8p)

(b) Estimate the current value of the delta of the option using your tree. . . . (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding.

(a) A stock is expected to pay a dividend of 5% of its value in three months time. The stock price is \$100, and the risk-free rate of interest is 3% per annum with continuous compounding. An investor has just taken a long position in a nine-month forward contract on the stock.

i. What are the forward price and the initial value of the forward contract? . . . . (3p)

ii. Consider a nine-month European put option written on the stock. The strike price of the option is 95, and the volatility of the stock is 30%. Compute the price of the European put option. . . . . (4p)

Hint: The volatility of the forward price is the same as the volatility of the stock price, and when interest rates are deterministic forward prices are equal to futures prices.

(b) The payoff at maturity T of a *cash-or-nothing call* is

$$X_{cn} = \begin{cases} K & \text{if } S_T > K, \\ 0 & \text{otherwise} \end{cases}$$

where K denotes a prespecified amount of cash. The price of a cash-or-nothing call written on a non-dividend paying stock is at time t

$$\Pi_t(X_{cn}) = e^{-r(T-t)} K \Phi[d_2(t, S_t)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution, S_t is the stock price at time t, and

$$d_2(t,s) = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) \right\}.$$

Compute delta for a cash-or-nothing call option. . . . . (3p)

4. Suppose that the following bonds are currently trading in the market:

- A zero coupon bond with face value 100 and maturity in six months trades at 99.0050
- A coupon bond with face value 100, maturity one year, and a coupon of 3% per annum paid semi-annually trades at 100.4790
- A coupon bond with face value 100, maturity in 18 months, and a coupon of 3% per annum paid annually trades at 101.4379

(a) Compute the current term structure, i.e. the zero rates for six months, one year, and 18 months. Quote the rates per annum with continuous compounding. (3p)

(b) Compute the one-year forward rate starting after six months. Quote the rate both per annum with continuous compounding AND as a simple rate. . . . (2p)

(c) The six-month forward rate starting after six months is 3% and the six-month forward rate starting after one year is 4%, both quoted using continuous compounding. State and prove (or give an arbitrage argument for) a relationship between the forward rate computed in Exercise 4b and the two just given. (2p)

(d) Compute the forward price of the coupon bond with maturity in 18 months to be delivered in one year. . . . . (3p)

5. For this exercise consider the standard Black-Scholes model.

(a) A *gap call option* is a T-claim X_{gc}, such that $X_{gc} = \phi(S_T)$, where the payoff function ϕ is defined by

$$\phi(s) = \begin{cases} s - K_s, & \text{if } s > K_t, \\ 0, & \text{otherwise.} \end{cases}$$

Here K_s (sub-index s for strike price) and K_t (sub-index t for trigger price) are non-negative constants. A *gap put option* is a T-claim X_{gp}, such that $X_{gp} = \psi(S_T)$, where the payoff function ψ is defined by

$$\psi(s) = \begin{cases} K_s - s, & \text{if } s < K_t, \\ 0, & \text{otherwise.} \end{cases}$$

i. Draw the payoff function for a gap call option and a gap put option for the case when $K_s \leq K_t$. . . . . (2p)

ii. The payoff functions are not piece-wise linear, but it is still possible to state a put-call parity for gap options. Do so! . . . . (3p)
Hint: The put-call parity looks the same for $K_s \leq K_t$ and $K_s > K_t$.

(b) Determine the arbitrage price of a *powered call*, which is a T-claim $X = \phi(S_T)$ with contract function ϕ given by

$$\phi(s) = \begin{cases} (s - K)^2, & \text{if } s > K \\ 0, & \text{otherwise.} \end{cases}$$

. . . . (5p)
Hint: On next page!

The payoff function can be written as $\phi(s) = (s-K)^2 I\{s > K\}$, where the notation $I\{s > K\}$ is used for the indicator function

$$I\{s > K\} = \begin{cases} 1, & \text{if } s > K \\ 0, & \text{otherwise.} \end{cases}$$

If you develop the square and compare this with the payoff of a standard call option $\phi_{call}(s) = (s-K)I\{s > K\}$ you will find that only one term will require work when pricing.

Good luck!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = C(t, S(t))$, where

$$C(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \leq x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$