Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI, EXAM IN SF2940 PROBABILITY THEORY TUESDAY THE 23^{rd} OF OCTOBER 2007 08.00 a.m.–01.00 p.m.

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six(6).

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Tuesday 23^{rd} of October 2007 at 01.05 p.m..

The exam results will be announced at the latest on Tuesday the 13^{th} of November on the announcement board of Matematisk statistik at the entry hall of Institutionen för matematik, Lindstedtsvägen 25.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

$X \in \mathrm{Po}(\lambda)$.

a) Show that $\psi(t) = E\left[e^{tX} \mid X > 0\right] = \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(e^{\lambda e^t} - 1\right)$. (7 p)

b) Find $E[X \mid X > 0]$ using $\psi(t)$ in part a). (3 p)

Uppgift 2

$X_1, X_2, \ldots,$ are I.I.D. random variables with $X_i \in U(0,1)$. We set

$$M_n = \max(X_1, X_2, \ldots, X_n),$$

and

$$U_n = n(1 - M_n).$$

What is the limit in distribution of U_n, as $n \to \infty$. Motivate your computations.

Hint: Consider $P(U_n > x)$ for $x > 0$ and its limit as $n \to \infty$. (10 p)

Uppgift 3

a) $\{X_n\}_{n\geq 1}^{\infty}$ is a sequence of random variables such that $E[X_n] = \mu_n$ and $\mathrm{Var}[X_n] = \sigma_n^2$, $n \geq 1$. Assume that

$$\lim_{n\to\infty} \mu_n = \mu, \quad \lim_{n\to\infty} \sigma_n^2 = 0.$$

Show that

$$X_n \xrightarrow{P} \mu, \quad \text{as } n \to \infty.$$

(7 p)

b) $X_n \in \Gamma\left(n, \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots,$. Find a number c such that

$$X_n \xrightarrow{d} c, \quad \text{as } n \to \infty$$

without using characteristic functions. (3 p)

Uppgift 4

$X \in U(0,1)$ and $Y \in U(0,1)$. X and Y are independent. Set $U = X + Y$ and $V = X - Y$.

a) Find the characteristic functions $\phi_U(t)$ and $\phi_V(t)$ of U and V, respectively. (4 p)

b) Show that U and V are not independent random variables. (6 p)

Uppgift 5

$X = \{X(t) \mid -\infty < t < \infty\}$ is a Gaussian stochastic process. Its mean function is $\mu(t) = 0$ for all t and its autocorrelation function is

$$E\left(X(t) \cdot X(s)\right) = R(h) = e^{-|h|}\cos(h), \quad h = t - s.$$

a) What is $E[X(t) \mid X(s)]$ for $t > s$? (2 p)

b) Please find the conditional probability

$$P\left(X(0.25) < 1 \mid X(0) = 1\right).$$

(8 p)

Uppgift 6

$W = \{W(t) \mid t \geq 0\}$ is a Wienerprocess. We form a new process $Y = \{Y(t) \mid t \geq 0\}$ with

$$Y(t) = W(2t) - W(t), \quad t \geq 0.$$

a) What is the probability distribution of $Y(t)$ for $t > 0$? (2 p)

b) Is Y a Wienerprocess ? Motivate your answer in detail. (8 p)

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SOLUTIONS TO THE EXAM IN SF2940 PROBABILITY THEORY 07-10-23

Uppgift 1

a) The conditional probability of X given B is denoted by $p_X(x \mid B)$ and given by (se the collection of formulas)

$$p_X\left(x_{|}B\right) = \begin{cases} \frac{p_X(x)}{p_X(B)} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

If $X \in \mathrm{Po}(\lambda)$, then $p_X(X > 0) = 1 - e^{-\lambda}$. Therefore

$$p_X(x \mid X > 0) = \begin{cases} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^k}{k!} & \text{if } x = k,\ k = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

This is a probability distribution and therefore

$$\psi(t) = E\left[e^{tX} \mid X > 0\right] = \sum_{k=1}^{\infty} e^{tk} p_X(k \mid X > 0) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=1}^{\infty} \frac{\left(e^t \lambda\right)^k}{k!}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(e^{e^t \lambda} - 1\right),$$

where we used the series expansion for e^x in the collection of formulas. This gives $\psi(t)$ as claimed.

b) $\psi(t) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(e^{\lambda e^t} - 1\right)$ is by construction the moment generating function of $X \mid X > 0$. Hence

$$E[X \mid X > 0] = \frac{d}{dt}\psi(t) \mid_{t=0}.$$

Then

$$\frac{d}{dt}\psi(t) = \frac{e^{-\lambda}}{1-e^{-\lambda}} e^{\lambda e^t} \lambda e^t$$

and

$$\frac{d}{dt}\psi(0) = \frac{e^{-\lambda}}{1-e^{-\lambda}} e^{\lambda} \lambda = \frac{\lambda}{1-e^{-\lambda}}$$

ANSWER b): $\underline{E[X \mid X > 0] = \frac{\lambda}{1-e^{-\lambda}}}$.

Uppgift 2

If $x \leq 0$, then $P(U_n > x) = 1$. Take $x > 0$.

$$P(U_n > x) = P(n(1 - M_n) > x) = P\left(M_n < 1 - \frac{x}{n}\right)$$

$$= P\left(\max(X_1, X_2, \ldots, X_n) < 1 - \frac{x}{n}\right)$$

But $\max(X_1, X_2, \ldots, X_n) < 1 - \frac{x}{n}$ if and only if all $X_i < 1 - \frac{x}{n}$. As $X_1, X_2, \ldots,$ are independent we get

$$= \prod_{i=1}^{n} P\left(X_i < 1 - \frac{x}{n}\right).$$

Because $X_i \in U(0,1)$, we have, since $\frac{x}{n} < 1$, that $P\left(X_i < 1 - \frac{x}{n}\right) = 1 - \frac{x}{n}$. Therefore

$$\prod_{i=1}^{n} P\left(X_i < 1 - \frac{x}{n}\right) = \left(1 - \frac{x}{n}\right)^n.$$

Hence we have shown for $x > 0$ that

$$P(U_n > x) = \left(1 - \frac{x}{n}\right)^n \Leftrightarrow P(U_n \leq x) = 1 - \left(1 - \frac{x}{n}\right)^n.$$

Then it follows that for $x > 0$

$$F_{U_n}(x) = P(U_n \leq x) = 1 - \left(1 - \frac{x}{n}\right)^n \to 1 - e^{-x}.$$

We have

$$F_{U_n}(x) = 0 \to 0 \qquad x \leq 0.$$

The limiting distribution function is thus

$$F(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and $F(x)$ is continuous for all x, hence $F_{U_n}(x) \to F(x)$ for all points of continuity of $F(x)$. The function $F(x)$ is the distribution function of Exp(1).

ANSWER: $\underline{U_n \xrightarrow{d} \text{Exp}(1).}$

Uppgift 3

a) We are going to establish first that $X_n \xrightarrow{q} \mu$, i.e., that

$$E\left[(X_n - \mu)^2\right] \to 0, \qquad \text{as } n \to \infty.$$

We have by some high school algebra that

$$E\left[(X_n - \mu)^2\right] = E\left[X_n^2\right] - 2\mu E\left[X_n^2\right] + \mu^2$$

$$= \text{Var}(X_n) + (E[X_n])^2 - 2\mu E\left[X_n^2\right] + \mu^2$$

$$= \sigma_n^2 + \mu_n^2 - 2\mu\mu_n + \mu^2$$

As $n \to \infty$, we get by assumptions in this assignment

$$E\left[(X_n - \mu)^2\right] = \sigma_n^2 + \mu_n^2 - 2\mu\mu_n + \mu^2$$

$$\to 0 + \mu^2 - 2\mu^2 + \mu^2 = 0,$$

which establishes mean square convergence. It is known that $X_n \xrightarrow{q} \mu \Rightarrow X_n \xrightarrow{P} \mu$, and the assertion follows as claimed.

b) By A. Gut, loc.cit, we get that if $X_n \in \Gamma\left(n, \frac{1}{n}\right)$, then

$$\mu_n = E\left[X_n\right] = n \cdot \frac{1}{n} = 1, \quad \sigma_n^2 = \mathrm{Var}\left[X_n\right] = n \cdot \frac{1}{n^2} = \frac{1}{n}.$$

Hence $\lim_{n\to\infty} \mu_n = 1$ and $\lim_{n\to\infty} \sigma_n^2 = 0$. From part a) of this Uppgift we get that $X_n \xrightarrow{P} \mu$, and this is known to imply that $X_n \xrightarrow{d} \mu$.

ANSWER b): $\underline{X_n \xrightarrow{d} \mu.}$

Uppgift 4

a) By definition of the characteristic function and by the assumed independence of X and Y we get

$$\phi_U(t) = E\left[e^{iU}\right] = E\left[e^{i(X+Y)}\right] = E\left[e^{iX}\right] E\left[e^{iY}\right]$$

$$= \phi_X(t) \cdot \phi_Y(t) = \left(\frac{e^{it} - 1}{it}\right)^2,$$

where we invoked A. Gut, loc.cit., too. For the same reasons we have

$$\phi_V(t) = E\left[e^{itV}\right] = E\left[e^{it(X-Y)}\right] = E\left[e^{itX}\right] E\left[e^{-itY}\right],$$

$$= \phi_X(t) \cdot \phi_Y(-t) = \left(\frac{e^{it} - 1}{it}\right)\left(\frac{e^{-it} - 1}{-it}\right)$$

$$= \left(\frac{\left(e^{it} - 1\right)\left(e^{-it} - 1\right)}{t^2}\right)$$

ANSWER a): $\underline{\phi_U(t) = \left(\frac{e^{it}-1}{it}\right)^2, \phi_V(t) = \left(\frac{\left(e^{it}-1\right)\left(e^{-it}-1\right)}{t^2}\right).}$

b) Let us consider the joint characteristic function

$$\phi_{U,V}(t_1, t_2) = E\left[e^{i(t_1 U + t_2 V)}\right]$$

For U and V to be independent, a necessary (and sufficient) condition is

$$\phi_{U,V}(t_1, t_2) = \phi_U(t_1) \cdot \phi_V(t_2)$$

for all t_1 and t_2. We have

$$\phi_{U,V}(t_1, t_2) = E\left[e^{i(t_1(X+Y) + t_2(X-Y))}\right]$$

$$= E\left[e^{i((t_1+t_2)X+(t_1-t_2)Y)}\right]$$
$$= E\left[e^{i((t_1+t_2)X)}\right] \cdot E\left[e^{i((t_1-t_2)Y)}\right]$$

by independence of X and Y, and

$$= \left(\frac{e^{i(t_1+t_2)}-1}{i(t_1+t_2)}\right)\left(\frac{e^{i(t_1-t_2)}-1}{i(t_1-t_2)}\right).$$

It follows with part a) that there are valuesof t_1 and t_2 such that

$$\phi_{U,V}(t_1,t_2) \neq \phi_U(t_1) \cdot \phi_V(t_2).$$

To check this, take, e.g., $t_1 = t_2 = \pi$. Hence U and V are not independent.

Uppgift 5

a) Since X is a Gaussian process, we find $E\left[X(t) \mid X(s)\right]$ for $t > s$ in the collection of formulas and by the Doob-Dynkin interpretation of conditional expectation. We have

$$E\left[X(t) \mid X(s)\right] = \mu_{X(t)} + \rho\frac{\sigma_{X(t)}}{\sigma_{X(s)}}X(s)$$

Here $\mu_{X(t)} = 0$ and

$$\sigma^2_{X(t)} = \sigma^2_{X(s)} = R(0) = e^{-|0|}\cos(0) = 1.$$

and since means are zero,

$$\rho = \frac{\mathrm{Cov}\left[X(t), X(s)\right]}{\sigma_{X(t)} \cdot \sigma_{X(s)}} = e^{-|t-s|}\cos(t-s)$$

Hence we have obtained

ANSWER a): $\underline{E\left[X(t) \mid X(s)\right] = e^{-|t-s|}\cos(t-s) \cdot X(s).}$

b) The conditional distribution of $X(0.25)$ given $X(0) = 1$ is

$$N\left(e^{-0.25}\cos(0.25), 1 - e^{-0.5}\cos^2(0.25)\right)$$

We set

$$Y = X(0.25) \mid X(0).$$

Then

$$P\left(X(0.25) < 1 \mid X(0) = 1\right) = P\left(\frac{Y - e^{-0.25}\cos(0.25)}{\sqrt{1 - e^{-0.5}\cos^2(0.25)}} < \frac{1 - e^{-0.25}\cos(0.25)}{\sqrt{1 - e^{-0.5}\cos^2(0.25)}}\right) =$$

$$= \Phi\left(\frac{1 - e^{-0.25}\cos(0.25)}{\sqrt{1 - e^{-0.5}\cos^2(0.25)}}\right).$$
$$= \Phi(0.374) = 1 - Q(0.374) \approx 0.65.$$

ANSWER b): $\underline{P\left(X(0.25) < 1 \mid X(0) = 1\right) = 0.65.}$

Uppgift 6

a) $Y(t)$ is a linear combination of two jointly normal random variables. Hence it is a normal random variable, or, it is an increment of a Wienerprocess and therefore a normal random variable. We need to find its mean and variance. By properties if a Wienerprocess we get

$$E\left[Y(t)\right] = E\left[W(2t)\right] - E\left[W(t)\right] = 0 + 0 = 0.$$

$$\begin{aligned}\mathrm{Var}\left[Y(t)\right] &= \mathrm{Var}\left[W(2t)\right] + \mathrm{Var}\left[W(t)\right] - 2\mathrm{Cov}\left(W(2t), W(t)\right) \\ &= 2t + t - 2\min(2t, t) = 2t + t - 2t = t.\end{aligned}$$

Hence we have

ANSWER a): $\underline{Y(t) \in N(0, t).}$

b) It is clear that Y is a Gaussian process. Its mean function is zero. We check the autocovariance (or, in fact, autocorrelation). For $t > s$ we get

$$E\left[Y(t) \cdot Y(s)\right] = E\left[(W(2t) - W(t))(W(2s) - W(s)\right]$$

$$\begin{aligned}&= E\left[(W(2t)W(2s)\right] - E\left[W(2t)W(s)\right] - E\left[W(t)W(2s)\right] + E\left[W(t)W(s)\right] \\ &= 2s - s - \min(t, 2s) + s = 2s - \min(t, 2s) \\ &= \begin{cases} 0 & \text{if } t > 2s \\ 2s + t & s < t < 2s. \end{cases}\end{aligned}$$

This is not the autocovariance of a Wienerprocess. In fact, the increments of Y are not independent.

ANSWER b): $\underline{Y \text{ is NOT a Wienerprocess.}}$