Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY TUESDAY THE 7^{th} OF JANUARY 2009 02.00 p.m.–07.00 p.m.

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six(6).

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Tuesday 7^{th} of January 2009 at 19.05 p.m..

The exam results will be announced at the latest on Tuesday the 20^{th} of January.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

AN ADDITION TO THE COLLECTION OF FORMULAS :
In one of the assignments (Uppgift 1-6) below the following fact (Markov's inequality) may turn out to be useful. If $X \geq 0$, then

$$P(X \geq \epsilon) \leq \frac{E[X]}{\epsilon}, \quad \epsilon > 0.$$

A CORRECTION TO THE COLLECTION OF FORMULAS, Section 2.4: The formula for Var(X) should read

$$\mathrm{Var}(X) = \mathrm{Var}(E(X|Y)) + E(\mathrm{Var}(X|Y)).$$

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Uppgift 1

The random variables $X_1, X_2, \ldots, X_{10}$ are such that

$$P(X_i \neq 0) \leq \frac{1}{1000}, \quad i = 1, 2, \ldots, 10.$$

Show that

$$P\left(\sum_{i=1}^{10} X_i \neq 0\right) \leq \frac{1}{100}.$$

(10 p)

Uppgift 2

The probability generating function $g(t)$ of a random variable X satisfies for all t in its domain of convergence the functional equation

$$g(t) = t \cdot e^{\lambda(g(t)-1)}, \quad 0 \leq \lambda < 1.$$

(a) Find $E[X]$. (3 p)

(b) Find $\mathrm{Var}[X]$. (7 p)

Uppgift 3

Let $X_1, X_2, \ldots,$ be a sequence of I.I.D. random variables such that $P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$. Let $Y_1, Y_2, \ldots,$ be another sequence of I.I.D. random variables such that $P(Y_i = 0) = P(Y_i = +1) = \frac{1}{2}$. Define for $n = 1, 2, \ldots$

$$G_n = \frac{\sum_{i=1}^{n} X_i \sqrt{\sum_{i=1}^{n} Y_i}}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i}.$$

Show that

$$G_n \xrightarrow{d} N(0, 2) \quad \text{as } n \to \infty.$$

(10 p)

Uppgift 4

$X_1, X_2, \ldots,$ is a sequence of random variables such that $X_0 > 0$, $E[X_0] < \infty$, and

$$X_n = X_{n-1}\left(\alpha + \beta Z_n^2\right), \quad n = 1, 2, \ldots,$$

where $Z_1, Z_2, \ldots,$ is a sequence of I.I.D. random variables with $Z_i \in N(0,1)$, and are independent of X_0. $\alpha > 0$ and $\beta > 0$ are constants.

(a) Let $\mathcal{F}_n = \sigma\{X_n, X_{n-1}, \ldots, X_1, X_0\}$ be the sigma field generated by the random variables up to time n. Is $(X_n, \mathcal{F}_n)_{n \geq 0}$ a martingale ? You must justify your answer. (2 p)

(b) Show that if $\alpha + \beta < 1$, then

$$X_n \xrightarrow{p} 0, \quad \text{as } n \to \infty.$$

(8 p)

Uppgift 5

$W = \{W(t) \mid 0 \leq t < \infty\}$ is a Wiener process. Set for $0 \leq t \leq 1$

$$W^o(t) = W(t) - t \cdot W(1).$$

Find the probability

$$P\left(W^o(0.4) > 0.5\right).$$

(10 p)

Uppgift 6

Let X_n be a random variable which has the uniform distribution on $\{0, 1, 2, \ldots, n-1\}$, i.e.,

$$P\left(X_n = k\right) = \frac{1}{n}, \quad k = 0, 1, 2 \ldots, n-1.$$

(a) Find the characteristic function $\varphi_{X_n}(t)$ of X_n. Your answer must be in a closed form, i.e., not, e.g., a sum of general nature. (2 p)

(b) Show using the characteristic functions $\varphi_{X_n}(t)$ that

$$\frac{X_n}{n} \xrightarrow{d} U(0,1), \quad \text{as } n \to \infty.$$

(3 p)

(c) Set

$$Y_n = \frac{X_n^2}{n^2}.$$

Show that there is a distribution function F so that

$$Y_n \xrightarrow{d} F \quad \text{as } n \to \infty,$$

and give the expression for F. (5 p)

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SOLUTIONS TO THE EXAM TUESDAY THE 7^{th} OF JANUARY 2009 02.00 p.m.–07.00 p.m. .

Uppgift 1

We set, for convenience of writing and clarity of argument,

$$A_i = \{X_i \neq 0\}.$$

We have that in case the event $\left\{\sum_{i=1}^{10} X_i \neq 0\right\}$ occurs, then at least one of the events A_i must have occurred. In other words

$$\left\{\sum_{i=1}^{10} X_i \neq 0\right\} \subseteq \bigcup_{i=1}^{10} A_i.$$

Therefore (c.f., the Collection of Formulas)

$$P\left(\sum_{i=1}^{10} X_i \neq 0\right) \leq P\left(\bigcup_{i=1}^{10} A_i\right).$$

By Boole$'$s inequality in the Collection of Formulas it holds that

$$P\left(A \cup B\right) \leq P(A) + P(B).$$

If we repeat this inequality a sufficient number of times we get for the case at hand

$$P\left(\bigcup_{i=1}^{10} A_i\right) \leq \sum_{i=1}^{10} P\left(A_i\right).$$

We know by the statement of the problem that

$$P\left(A_i\right) \leq \frac{1}{1000}.$$

Hence we have by the preceding that

$$P\left(\sum_{i=1}^{10} X_i \neq 0\right) \leq \sum_{i=1}^{10} \frac{1}{1000} = \frac{10}{1000} = \frac{1}{100}$$

as claimed.

Uppgift 2

(a) In order to find $E\left[X\right]$ we differentiate the p.g.f. once w.r.t. t. Denoting the first derivative by $g^{'}(t)$ we have

$$g^{'}(t) = e^{\lambda(g(t)-1)} + t \cdot e^{\lambda(g(t)-1)} \lambda g^{'}(t).$$

We put $t = 1$ and recall $g(1) = 1$ to obtain

$$g^{'}(1) = 1 + \lambda g^{'}(t).$$

This gives

$$E\left[X\right] = g^{'}(1) = \frac{1}{1-\lambda}.$$

ANSWER (a): $\underline{E\left[X\right] = \frac{1}{1-\lambda}}$.

(b) We differentiate once more to obtain

$$g^{''}(t) = e^{\lambda(g(t)-1)}\lambda g^{'}(t) + e^{\lambda(g(t)-1)}\lambda g^{'}(t) + t\cdot e^{\lambda(g(t)-1)}\lambda^2 g^{'}(t)^2 + t\cdot e^{\lambda(g(t)-1)}\lambda g^{''}(t).$$

Again we put $t = 1$ and recall $g(1) = 1$ to obtain

$$g^{''}(1) = \lambda g^{'}(1) + \lambda g^{'}(1) + \lambda^2 g^{'}(1)^2 + \lambda g^{''}(1).$$

We insert by part (a) of this Uppgift $g^{'}(1) = \frac{1}{1-\lambda}$ in the expression above

$$g^{''}(1) = 2\lambda\frac{1}{1-\lambda} + \lambda^2\left(\frac{1}{1-\lambda}\right)^2 + \lambda g^{''}(1).$$

A simple piece of high school algebra yields

$$g^{''}(1) = \lambda\frac{2-\lambda}{(1-\lambda)^3}.$$

Next, the Collection of Formulas, section 7.1.2., states that

$$\begin{aligned} g^{''}(1) &= E\left[X(X-1)\right] \\ &= E\left[X^2\right] - E\left[X\right], \end{aligned}$$

i.e.,

$$E\left[X^2\right] = g^{''}(1) + E\left[X\right] = \lambda\frac{2-\lambda}{(1-\lambda)^3} + \frac{1}{1-\lambda}.$$

Then the variance σ^2 is given by

$$\begin{aligned} \sigma^2 &= E\left[X^2\right] - \left(E\left[X\right]\right)^2 \\ &= \lambda\,\frac{2-\lambda}{(1-\lambda)^3} + \frac{1}{1-\lambda} - \left(\frac{1}{1-\lambda}\right)^2. \end{aligned}$$

From here some straightforward algebra gives

$$\sigma^2 = \frac{\lambda}{(1-\lambda)^3}.$$

ANSWER (b): $\underline{\sigma^2 = \frac{\lambda}{(1-\lambda)^3}}$.

Uppgift 3

Let $X_1, X_2, \ldots,$ be a sequence of I.I.D. random variables such that $P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$. Let $Y_1, Y_2, \ldots,$ be another sequence of I.I.D. random variables such that $P(Y_i = 0) = P(Y_i = +1) = \frac{1}{2}$.
Hence it holds for all i that

$$E[X_i] = 0, E[Y_i] = \frac{1}{2}.$$

Furthermore it holds for all i that

$$\text{Var}[X_i] = 1.$$

The central limit theorem implies now that as $n \to \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} N(0,1)$$

The law of large numbers implies that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} 0,$$

and

$$\frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{p} \frac{1}{2},$$

as $n \to \infty$.
Now we can write

$$G_n = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \sqrt{\frac{1}{n} \sum_{i=1}^{n} Y_i}}{\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} Y_i}.$$

Since $g(x) = \sqrt{x}$ is a continuous function at any point $x \geq 0$, we get

$$\sqrt{\frac{1}{n} \sum_{i=1}^{n} Y_i} \xrightarrow{p} \sqrt{\frac{1}{2}}.$$

We have by the preceding and the basic properties of convergence in probabability that

$$\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{p} \frac{1}{2}.$$

Hence we obtain by the Cramér-Slutzky theorem

$$G_n \xrightarrow{d} \frac{X \cdot \sqrt{\frac{1}{2}}}{\frac{1}{2}} = \frac{X}{\sqrt{\frac{1}{2}}} = \sqrt{2}X,$$

where $X \in N(0,1)$. Hence $\sqrt{2}X \in N(0,2)$, as claimed.

Uppgift 4

(a) We check first formally the martingale property.

$$E\left[X_n \mid \mathcal{F}_{n-1}\right] = E\left[X_{n-1}\left(\alpha+\beta Z_n^2\right) \mid \mathcal{F}_{n-1}\right] = X_{n-1}E\left[\left(\alpha+\beta Z_n^2\right) \mid \mathcal{F}_{n-1}\right]$$

as X_{n-1} is measurable w.r.t. $\mathcal{F}_{n-1}$. Because $Z_1, Z_2, \ldots,$ is a sequence of I.I.D. random variables independent of X_0, Z_n is independent of $\mathcal{F}_{n-1}$ and

$$E\left[\left(\alpha+\beta Z_n^2\right) \mid \mathcal{F}_{n-1}\right] = E\left[\left(\alpha+\beta Z_n^2\right)\right] = \alpha+\beta E\left[Z_n^2\right] = \alpha+\beta.$$

Thereby we have obtained

$$E\left[X_n \mid \mathcal{F}_{n-1}\right] = X_{n-1}\left(\alpha+\beta\right).$$

Therefore the martingale property is satisfied in and only if $\alpha+\beta=1$. It is straightforward to check adaptedness and integrability for all feasible values of α and β.

ANSWER: $\left(X_n, \mathcal{F}_n\right)_{n\geq 0}$ is a martingale if $\alpha+\beta=1$.

(b) We are required to demonstrate that for any $\epsilon > 0$

$$P\left(\mid X_n \mid \geq \epsilon\right) \to 0, \quad \text{as } n \to \infty.$$

First we observe that $X_n > 0$. This follows by iteration as

$$X_n = X_{n-1}\left(\alpha+\beta Z_n^2\right) = X_{n-2}\left(\alpha+\beta Z_{n-1}^2\right)\left(\alpha+\beta Z_n^2\right)$$

$$= \ldots = X_0\left(\alpha+\beta Z_1^2\right)\cdot\ldots\cdot\left(\alpha+\beta Z_n^2\right)$$

Since $X_0 > 0$ and $\alpha > 0$ and $\beta > 0$, then $X_n > 0$. We apply now Markov's inequality to the effect that for any $\epsilon > 0$

$$P\left(\mid X_n \mid \geq \epsilon\right) = P\left(X_n \geq \epsilon\right) \leq \frac{E\left[X_n\right]}{\epsilon}, \quad \epsilon > 0.$$

Since $Z_1, Z_2, \ldots,$ is a sequence of I.I.D. random variables independent of X_0, and $E\left[X_0\right] < \infty$, we have that

$$E\left[X_n\right] = E\left[X_0\right]E\left[\left(\alpha+\beta Z_1^2\right)\cdot\ldots\cdot\left(\alpha+\beta Z_n^2\right)\right] = E\left[X_0\right]E\left[\alpha+\beta Z_1^2\right]\cdot\ldots\cdot E\left[\alpha+\beta Z_n^2\right]$$

$$= E\left[X_0\right]\left(\alpha+\beta\right)^n.$$

Thus we have

$$P\left(\mid X_n \mid \geq \epsilon\right) \leq \frac{E\left[X_0\right]\left(\alpha+\beta\right)^n}{\epsilon}$$

and the desired conclusion follows, since $\left(\alpha+\beta\right)^n \to 0$, as $n \to \infty$, when $0 < \alpha+\beta < 1$.

Uppgift 5

We shall invoke a trick (the problem can be solved without this) by writing

$$W^o(t) = W(t) - t\cdot W(1) = (1-t)W(t) - t\cdot\left(W(1)-W(t)\right).$$

By the properties of the Wiener process W we have

$$W(t) \in N(0,t), \quad W(1) - W(t) \in N(0, 1-t),$$

as well as that $W(t)$ and $W(1) - W(t)$ are independent.
Then we have

$$E\left[W^o(t)\right] = (1-t)E\left[W(t)\right] - t \cdot E\left[W(1) - W(t)\right] = 0.$$

and by independence of $W(t)$ and $W(1) - W(t)$

$$\mathrm{Var}\left[W^o(t)\right] = (1-t)^2 t + t^2(1-t) = t - 2t^2 + t^3 + t^2 - t^3 = t \cdot (1-t).$$

Thus we have

$$W^o(t) \in N(0, t(1-t)).$$

Hence we know that

$$\frac{W^o(0.4)}{\sqrt{0.4 \cdot 0.6}} \in N(0,1).$$

The sought probability is therefore

$$P\left(W^o(0.4) > 0.5\right) = P\left(\frac{W^o(0.4)}{\sqrt{0.4 \cdot 0.6}} > \frac{0.5}{\sqrt{0.4 \cdot 0.6}}\right) = 1 - \Phi\left(\frac{0.5}{\sqrt{0.24}}\right) \approx 0.154,$$

where Φ is the distribution function of the standard normal distribution.

$$\text{ANSWER} : \underline{P\left(W^o(0.4) > 0.5\right) = \approx 0.154.}$$

Uppgift 6

(a) The characteristic function is by definition

$$\varphi_{X_n}(t) = E\left[e^{itX_n}\right] = \sum_{k=0}^{n-1} e^{itk}\frac{1}{n} = \frac{1}{n}\sum_{k=0}^{n-1}\left(e^{it}\right)^k =$$

$$= \frac{1}{n}\frac{1 - e^{itn}}{1 - e^{it}}$$

where we used the formula for a finite geometric sum (see section 12.2. in the Collection of Formulas).

$$\text{ANSWER (a)} : \underline{\varphi_{X_n}(t) = \tfrac{1}{n}\tfrac{1-e^{itn}}{1-e^{it}}.}$$

(b) By straightforward standard properties of characteristic functions we have

$$\varphi_{\frac{X_n}{n}}(t) = \varphi_{X_n}(t/n) = \frac{1}{n}\frac{1 - e^{it}}{1 - e^{it/n}}$$

We make a series expansion (see section 12.2. in the Collection of Formulas) in the denominator to get

$$n\left(1-e^{it/n}\right)=n\left(1-\left(1+\frac{it}{n}+O\left(\frac{|t|}{n^2}\right)\right)\right)=-it+O\left(\frac{|t|}{n}\right).$$

Thus, as $n\to\infty$,

$$\frac{1}{n}\frac{1-e^{it}}{1-e^{it/n}}=\frac{1-e^{it}}{-it+O\left(\frac{|t|}{n}\right)}\to\frac{1-e^{it}}{-it}=\frac{e^{it}-1}{it}.$$

The expression in the right hand side is recognized as the characteristic function of the uniform distribution $U(0,1)$. Hence the asserted convergence

$$\frac{X_n}{n}\xrightarrow{d}U(0,1),\qquad \text{as } n\to\infty$$

has been established as desired.

(c) By our design the random variable

$$Y_n=\frac{X_n^2}{n^2}.$$

has the uniform distribution on the set of points

$$0,\left(\frac{1}{n}\right)^2,\left(\frac{2}{n}\right)^2,\ldots,\left(\frac{n-1}{n}\right)^2.$$

Let now $0<y<1$. Then

$$P\left(Y_n\le y\right)=\frac{1}{n}\left(\text{number of } k \text{ such that } \left(\frac{k}{n}\right)^2\le y\right)$$

$$=\frac{1}{n}(\text{ number of } k \text{ such that } k\le\sqrt{y}n)=\frac{\lfloor\sqrt{y}n\rfloor}{n},$$

where $\lfloor\sqrt{y}n\rfloor$ is the integer part of $\sqrt{y}n$, i.e., the largest integer smaller than or equal to $\sqrt{y}n$. We write

$$\lfloor\sqrt{y}n\rfloor=\sqrt{y}n-\ll\sqrt{y}n\gg,$$

where the fractional part $\ll\sqrt{y}n\gg$ must satisfy $0\le\ll\sqrt{y}n\gg\le 1$. Thus

$$\frac{\lfloor\sqrt{y}n\rfloor}{n}=\sqrt{y}-\frac{\ll\sqrt{y}n\gg}{n}$$

and clearly, as $n\to\infty$,

$$\frac{\ll\sqrt{y}n\gg}{n}\le\frac{1}{n}\to 0.$$

It is obvious by the preceding that

$$P\left(Y_n\le y\right)=0,\quad y<0,$$

and

$$P(Y_n \leq y) = 1, \quad 1 \leq y,$$

Let us set

$$F(y) = \begin{cases} 0 & \text{if } y < 0, \\ \sqrt{y} & \text{if } 0 \leq y < 1, \\ 1 & \text{if } y \geq 1. \end{cases}$$

This satisfies obviously the properties defining a distribution function of a random variable. $F(y)$ is a continuous function for all y, thus we have established convergence

$$P(Y_n \leq y) \to F(y)$$

in every point of continuity of $F(y)$. Hence we have by definition of convergence in distribution shown that

$$Y_n \xrightarrow{d} F \quad \text{as } n \to \infty,$$

with the expression for F given above.