TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY THE 21^{st} OF OCTOBER 2009 08.00 a.m.–1.00 p.m.

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six(6).

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with $27-29$ points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 21^{st} of October 2009 at 1.05 p.m..

The exam results will be announced at the latest on Friday the 30^{th} of October.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

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Uppgift 1

The pair of random variables (X, Y) has the joint probability density given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2 + \left(y - \sqrt{|x|}\right)^2\right)}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

(a) Find a function $g(X)$ of the random variable X such that

$$E\left[(Y - g(X))^2\right]$$

is minimized. You are required to recapitulate in detail how you obtained $g(X)$. (8 p)

(b) Let $g^*(X)$ be the minimizing function established in (a). Compute the value of the optimal mean square error

$$E\left[(Y - g^*(X))^2\right].$$

(2 p)

Uppgift 2

Let the random variables Z_i, $i = 1, 2, \ldots$ be independent and identically distributed (I.I.D.) with the probability function $(0 < p < 1)$

$$p_{Z_i}(z) = p \cdot (1-p)^z, \quad z = 0, 1, 2, \ldots$$

We consider N random variables Z_1,Z_2, $\ldots$, Z_N, where N is a random variable, which is Po(λ)-distributed. N is independent of all Z_i:s. We set

$$S_N = Z_1 + Z_2 + \ldots + Z_N, \quad S_0 = 0.$$

Determine $P\left(S_N \geq 2\right)$. Show your calculations. (10 p)

Uppgift 3

$W = \{W(t) \mid 0 \leq t < \infty\}$ is a Wiener process. We set for $h > 0$

$$X_n = \sum_{k=1}^{n} \frac{1}{k} \left(W(hk) - W(h(k-1))\right), \quad n \geq 1.$$

(a) Determine the distribution of X_n. Please explain the steps of your solution in detail. (5 p)

(b) Show that there is a convergence in distribution

$$X_n \xrightarrow{d} X \quad \text{as } n \to \infty,$$

and find the distribution of the limiting random variable X. (5 p)

Uppgift 4

Let $Z_1, Z_2, \ldots,$ be a sequence of I.I.D. random variables such that $P\left(Z_i = +1\right) = p$, $P\left(Z_i = -1\right) = 1 - p$, $0 < p < 1$. Let

$$S_n = \sum_{i=1}^{n} Z_i, \quad n \geq 1,$$

and set for a real number M

$$X_n = M^{S_n}, n \geq 1.$$

Determine $M \neq 1$ such that $\left(X_n, \mathcal{F}_n\right)_{n \geq 1}$ satisfies the martingale property, where $\mathcal{F}_n = \sigma\left(Z_1, Z_2, \ldots, Z_n\right)$ is the sigma-field generated by $Z_1, Z_2, \ldots, Z_n$. (10 p)

Uppgift 5

$N = \{N(t) \mid t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$. Let T be the time of occurrence of the first event. Show that $T \mid N(t) = 1 \in U(0, t)$.

Hint: It may be a good idea to start with

$$P\left(T \leq x \mid N(t) = 1\right), \quad 0 \leq x \leq t.$$

(10 p)

Uppgift 6

Show that

$$\lim_{n \to \infty} \frac{1}{(n-1)!} \int_0^n x^{n-1} e^{-x} dx = \frac{1}{2}.$$

Hint: You may find it helpful to consider the central limit theorem for a suitable random variable with a Gamma distribution. (10 p)

SOLUTIONS TO THE EXAM WEDNESDAY THE 21^{st} OF OCTOBER 2009 08.00 a.m.–1.00 p.m..

Uppgift 1

(a) The Collection of Formulas gives

$$E\left[(Y-g(X))^2\right] = E\left[\mathrm{Var}\left[Y|X\right]\right] + E\left[(E\left[Y|X\right]-g(X))^2\right]. \tag{1}$$

This shows that if we take

$$g^*(X) = E\left[Y|X\right]$$

we minimize the targeted expression. We need to find $f_{Y|X}(y)$. We have that

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\sqrt{|x|})^2}dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$

since $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sqrt{|x|})^2}$ is the density of $N(\sqrt{\mid x \mid}, 1)$. Hence

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sqrt{|x|})^2},$$

and we get that

$$Y|X = x \in N(\sqrt{\mid x \mid}, 1). \tag{2}$$

Hence we get $E\left[Y|X\right] = \sqrt{\mid X \mid}$.

ANSWER (a): $\underline{g^*(X) = \sqrt{\mid X \mid}.}$

(b) In view of (2) we have that

$$\mathrm{Var}\left[Y|X\right] = 1.$$

Hence by (1)

$$E\left[(Y-g^*(X))^2\right] = E\left[\mathrm{Var}\left[Y|X\right]\right] = 1. \tag{3}$$

ANSWER (b): $\underline{E\left[(Y-g^*(X))^2\right] = 1.}$

Remark: $f_{X,Y}(x,y)$ is the density of Joakim Lübeck's *Valentine distribution*.
The web page `http://www.maths.lth.se/matstat/staff/joa/` elucidates the name assigned to the distribution.

Uppgift 2

We have that

$$P(S_N \geq 2) = 1 - P(S_N = 0) - P(S_N = 1),$$

and invoke the appropriate probability generating function (p.g.f.) to compute the two probabilities in the right hand side. If $g_N(t)$ is the p.g.f. of N, and $g_{Z_i}(t) = g_Z(t)$ is the p.g.f. of the I.I.D. Z_i's, then the p.g.f of S_N is

$$g_{S_N}(t) = g_N(g_Z(t)). \tag{4}$$

By definition we have that

$$g_{Z_i}(t) = E\left[t^{Z_i}\right] = \sum_{z=0}^{\infty} t^z p_{Z_i}(z)$$

$$= p\sum_{z=0}^{\infty} t^z \cdot (1-p)^z = \frac{p}{1-(1-p)t} = \frac{p}{1-qt}, \quad q = 1-p,$$

where we summed a geometric series, which is convergent if and only if $\mid qt \mid < 1$. These p.g.f.s are the same for all Z_i, so we set $g_{Z_i}(t) = g_Z(t)$. Next we need the p.g.f. for

$$S_N = Z_1 + \ldots + Z_N.$$

The Collection of Formulas gives $g_N(t) = e^{\lambda(t-1)}$, and this yields in (4)

$$g_{S_N}(t) = g_N\left(\frac{p}{1-qt}\right) = e^{\lambda(\frac{p}{1-qt}-1)}, \quad \mid qt \mid < 1.$$

Now we can employ the p.g.f $g_{S_N}(t)$ to the effect that $P(S_N = 0) = g_{S_N}(0)$, $P(S_N = 1) = \frac{d}{dt}g_{S_N}(0)$. Note that the p.g.f is defined and differentiable in a neighborhood of $t = 0$. Now we have

$$g_{S_N}(0) = e^{-\lambda q}.$$

Next we have

$$\frac{d}{dt}g_{S_N}(t) = \lambda e^{\lambda\left(\frac{p}{1-qt}-1\right)}\left(\frac{pq}{(1-qt)^2}\right).$$

Evaluating this at $t = 0$ we get

$$\frac{d}{dt}g_{S_N}(0) = \lambda pq e^{-\lambda q}.$$

ANSWER : $\underline{P(S_N \geq 2) = 1 - e^{-\lambda q}(1 + \lambda pq).}$

Uppgift 3

(a) If $W = \{W(t) \mid 0 \leq t < \infty\}$ is a Wiener process, then

$$Y_k = W(hk) - W(h(k-1) \in N(0,h)$$

for all k and the $Y_1, Y_2, \ldots, Y_n$ are independent, as the increments of a Wiener process are independent. Thus, as a linear combination of normal random variables,

$$X_n = \sum_{k=1}^{n} \frac{1}{k} Y_k$$

is a normal random variable. We need to find its expectation and variance. We have

$$E[X_n] = \sum_{k=1}^{n} \frac{1}{k} E[Y_k] = 0.$$

and

$$\mathrm{Var}(X_n) = \sum_{k=1}^{n} \frac{1}{k^2} \mathrm{Var}(Y_k) = h \sum_{k=1}^{n} \frac{1}{k^2},$$

where we used the fact that $Y_1, Y_2, \ldots, Y_n$ are independent and have the variance h.

ANSWER (a): $\underline{X_n \in N\left(0, h \sum_{k=1}^{n} \frac{1}{k^2}\right).}$

(b) By part (a) we have that the characteristic function of X_n is

$$\varphi_{X_n}(t) = e^{-\frac{t^2}{2} h \sum_{k=1}^{n} \frac{1}{k^2}}.$$

Recalling suitable courses in mathematics we know that the sum $\sum_{k=1}^{n} \frac{1}{k^2}$ converges, as $n \to \infty$, and

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{5}$$

Thus we get that

$$\varphi_{X_n}(t) \to \varphi_X(t) = e^{-\frac{t^2}{2} \frac{h\pi^2}{6}}, \quad n \to \infty.$$

The limiting function $\varphi_X(t)$ is the characteristic function of $N\left(0, \frac{h\pi^2}{6}\right)$. Thus the convergence above implies that

$$X_n \xrightarrow{d} N\left(0, \frac{h\pi^2}{6}\right)$$

as $n \to \infty$.

ANSWER (b): $\underline{X_n \xrightarrow{d} N\left(0, \frac{h\pi^2}{6}\right), \text{ as } \quad n \to \infty.}$

A remark on grading: there will not be a reduction of points for not remembering correctly the limit $\frac{\pi^2}{6}$ in (5), it suffices to know that the sum $\sum_{k=1}^{n} \frac{1}{k^2}$ converges as n grows to infinity.

Uppgift 4

We want to find M so that the martingale property, i.e.,

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n$$

holds for all $n \geq 1$. We start expanding in the left hand side above.

$$E\left[X_{n+1} \mid \mathcal{F}_n\right] = E\left[M^{S_{n+1}} \mid \mathcal{F}_n\right] = E\left[M^{S_n+Z_{n+1}} \mid \mathcal{F}_n\right]$$

$$= M^{S_n} E\left[M^{Z_{n+1}} \mid \mathcal{F}_n\right]$$

as S_n is by construction measurable w.r.t. $\mathcal{F}_n$. Because $Z_1, Z_2, \ldots,$ is a sequence of I.I.D. random variables, Z_{n+1} is independent of $\mathcal{F}_n$ (as it is independent of $Z_1, Z_2, \ldots, Z_n$), and hence

$$= M^{S_n} E\left[M^{Z_{n+1}}\right].$$

Thereby we have obtained

$$E\left[X_{n+1} \mid \mathcal{F}_n\right] = X_n E\left[M^{Z_{n+1}}\right].$$

Therefore the martingale property is satisfied if and only if $E\left[M^{Z_{n+1}}\right] = 1$. Now we have

$$E\left[M^{Z_{n+1}}\right] = pM + (1-p)M^{-1}.$$

Hence the question is reduced to solving w.r.t. M the equation

$$pM + (1-p)M^{-1} = 1.$$

We get

$$pM + (1-p)M^{-1} = 1 \Leftrightarrow pM^2 + (1-p) = M \Leftrightarrow pM^2 - M + (1-p) = 0.$$

The quadratic equation has two solutions

$$M = \begin{cases} \frac{1-p}{p} \\ 1. \end{cases}$$

The solution $M = 1$ is the trivial solution, i.e., it gives a constant martingale $X_n = 1$ for all n, which is evident from the outset. Also, if $p = 1/2$, we get $M = 1$. Hence for $p \neq 1/2$ we get that

$$\left(\left(\frac{1-p}{p}\right)^{S_n}, \mathcal{F}_n\right)_{n \geq 1}$$

satisfies the martingale property. It is in fact a martingale, known sometimes as *De Moivre's martingale*, but the question as stated is only about the martingale property.

$$\text{ANSWER: } \underline{M = \tfrac{1-p}{p},\ p \neq 1/2.}$$

Uppgift 5

Let us follow the given hint. It is clear by definition of T that $P(T \leq x \mid N(t) = 1) = 0$ if $x < 0$ and $P(T \leq x \mid N(t) = 1) = 1$ if $x > t$.
So take $0 \leq x \leq t$, and observe that

$$P(T \leq x \mid N(t) = 1) = \frac{P(N(x) = 1, N(t) = 1)}{P(N(t) = 1)},$$

since $\{T \leq x\} = \{N(x) \geq 1\}$, but as $N(t) = 1$, we have $N(x) = 1$ for $x \leq t$. This gives also

$$\frac{P(N(x)=1, N(t)=1)}{P(N(t)=1)} = \frac{P(N(x)=1, N(t)-N(x)=0)}{P(N(t)=1)}$$
$$= \frac{P(N(x)=1)\,P(N(t)-N(x)=0)}{P(N(t)=1)},$$

as the increment $N(t) - N(x)$ is independent of $N(x) = N(x) - N(0)$. Since N is a Poisson process with intensity $\lambda > 0$ we get

$$P(N(t)-N(x)=0) = e^{-\lambda(t-x)},\, P(N(x)=1) = \lambda x e^{-\lambda x},\, P(N(t)=1) = \lambda t e^{-\lambda t}.$$

When we insert these in the last expression we get

$$\frac{P(N(x)=1)\,P(N(t)-N(x)=0)}{P(N(t)=1)} = \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} = \frac{x}{t}.$$

Hence we have shown that for $0 \leq x \leq t$ we have the distribution function

$$F_{T|N(t)=1}(x) = P(T \leq x \mid N(t)=1) = \frac{x}{t}$$

or, if the probability density should seem more familiar,

$$f_{T|N(t)=1}(x) = \frac{d}{dx} F_{T|N(t)=1}(x) = \frac{1}{t}, \quad 0 \leq x \leq t,$$

and is zero elsewhere. Hence we have established that $f_{T|N(t)=1}(x)$ is the probability density of a random variable with the distribution $U(0,t)$, which establishes the assertion.

Uppgift 6

As hinted, let us take $X \in \Gamma(n,1)$, since then we have that

$$\frac{1}{(n-1)!}\int_0^n x^{n-1}e^{-x}dx = P(X \leq n),$$

after we have recognized the integrand in the left hand side as the density of $X \in \Gamma(n,1)$, found in the Appendix 2. Now we write

$$P(X \leq n) = P(X - n \leq 0) = P\left(\frac{X-n}{\sqrt{n}} \leq 0\right).$$

Since $X \stackrel{d}{=} U_1 + U_2 + \ldots + U_n$, where U_is are independent Exp(1) distributed random variables and thus $E[U] = 1$, $\mathrm{Var}(U) = 1$. Hence $E[X] = n$ and $\mathrm{Var}(X) = n$ (see also Appendix 2), and we have by the Central Limit Theorem

$$\frac{X-n}{\sqrt{n}} \stackrel{d}{\to} N(0,1), \qquad n \to \infty.$$

Therefore, by the very definition of convergence in distribution,

$$\lim_{n\to\infty} P\left(\frac{X-n}{\sqrt{n}} \leq 0\right) = \lim_{n\to\infty} F_{\frac{X-n}{\sqrt{n}}}(0) = \Phi(0),$$

where $\Phi(x)$ is the distribution function of $N(0,1)$ and as zero is one of its points of continuity (as are all points). But we know that $\Phi(0) = \frac{1}{2}$, and therefore we have established the desired limit as claimed.