Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY THE 12^{th} OF JANUARY 2011 2.00 p.m.–7.00 p.m.

Examinator: Timo Koski, tel. 790 71 34, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six(6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 12^{th} of January 2011 at 7.05 p.m..

The exam results will be announced at the latest on Friday the 28^{th} of January 2011.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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In one of the assignments You may find the following trigonometric identity helpful:

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Uppgift 1

$X \in N\left(v\cos(\phi), \sigma^2\right)$, $Y \in N\left(v\sin(\phi), \sigma^2\right)$, where X and Y are independent. Set

$$R = \sqrt{X^2 + Y^2}.$$

Show that the probability density $f_R(r)$ of R is

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{\left(r^2+v^2\right)}{2\sigma^2}} I_0\left(\frac{rv}{\sigma^2}\right),$$

where $I_0(x)$ is a modified Bessel function of the first kind with order 0. Note that

$$I_0\left(x\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos(\theta)} d\theta$$

Which distribution is obtained for $v = 0$? (10 p)

Hint: Introduce the additional random variable

$$\Theta = \arctan\left(\frac{Y}{X}\right), \quad -\pi \leq \Theta \leq \pi,$$

i.e., express X and Y in polar coordinates and find $f_{R,\Theta}(r, \theta)$.

Uppgift 2

Let X_i, $i = 1, 2, \ldots$, be I.I.D. random variables with the probability function

$$p_X(k) = -\frac{p^k}{k\ln(1-p)}, \quad k = 1, 2, \ldots, \quad 0 < p < 1.$$

Here ln is the natural logarithm. Let $N \in \mathrm{Po}\left(-n \cdot \ln(1-p)\right)$, n is a positive integer. The random variable N is independent of X_i, $i = 1, 2, \ldots,$.
We define the random variable S by

$$S \stackrel{\text{def}}{=} \sum_{i=1}^{N} X_i, \quad S = 0 \text{ if } N = 0.$$

Show that S is a random variable with negative binomial distribution, more precisely,

$$S \in \mathrm{NBin}(n, 1-p).$$

(10 p)

Uppgift 3

$$\begin{pmatrix} X \\ Y \end{pmatrix} \in N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right).$$

Let $g(y)$ be a Borel function such that $E\left[|g(Y)|\right] < \infty$. Show that

$$\mathrm{Cov}\left(X, g(Y)\right) = \frac{\mathrm{Cov}\left(Y, g(Y)\right)}{\sigma_2^2} \cdot \mathrm{Cov}\left(X, Y\right).$$

(10 p)

Uppgift 4

$\{W(t) \mid t \geq 0\}$ is a Wiener process. We define a new process $\mathbf{X} = \{X(t) \mid 0 \leq t < 1\}$ by the Wiener integral

$$X(t) \stackrel{\text{def}}{=} \int_0^t s dW(s).$$

Find a function $\tau(t)$ such that

$$X(t) \stackrel{d}{=} W(\tau(t)).$$

You can assume that $\tau(t)$ is strictly (mononously) increasing. You are expected to clarify in detail why Your solution works. (10 p)

Uppgift 5

The random variables $\{X_k\}_{k=0}^{\infty}$ are independent, and $X_k \in \mathrm{Po}\left(\frac{1}{k}\right)$ for every $k \geq 1$. We set

$$S_n = \sum_{k=1}^{n} k \cdot X_k + f(n), \quad S_0 = 0, f(0) = 0,$$

where $f(n)$ is a real valued function of the nonnegative integers, and does not depend on any random event. Determine $f(n)$ as a function of n so that $\{S_n\}_{n=1}^{\infty}$ has the martingale property w.r.t. the filtration (the increasing family of sigma-fields) $\{\sigma\left(X_0, X_1, \ldots, X_n\right)\}_{n=0}^{\infty}$. In other words $f(n)$ must be such that for $n \geq 1$

$$E\left[S_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = S_{n-1}$$

holds. Be so kind and explain Your steps of solution carefully. (10 p)

Uppgift 6

$X_1, X_2, \ldots,$ are I.I.D. random variables with the density

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-\frac{1}{2}(x-a)} & \text{if } x \geq a \\ 0 & \text{if } x < a, \end{cases}$$

where $a > 0$. We set

$$S_n = \sum_{i=1}^{n} X_i.$$

Show that

$$\frac{S_n - n(a+2)}{\sqrt{S_n}} \xrightarrow{d} N\left(0, m^2\right)$$

as $n \to \infty$ and determine the value of m^2. (10 p)

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SOLUTIONS TO THE EXAM WEDNESDAY THE 12^{th} OF JANUARY 2011 02.00 p.m.–07.00 p.m..

Uppgift 1

We set $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ and get

$$\frac{x}{r} = \cos(\theta), \frac{y}{r} = \sin(\theta).$$

The Jacobian $J = \mid \frac{\partial(x,y)}{\partial(r,\theta)} \mid$ is $J = r$. Then the change of variables formula in section 1.2 of *Formulas for probability theory* yields

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos(\theta), r\sin(\theta)) \cdot r$$
$$= \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r\cos(\theta)-v\cos(\phi))^2} e^{-\frac{1}{2\sigma^2}(r\sin(\theta)-v\sin(\phi))^2}.$$

We need to work out the exponent in the right hand side.

$$(r\cos(\theta) - v\cos(\phi))^2 + (r\sin(\theta) - v\sin(\phi))^2$$
$$= r^2\cos^2(\theta) - 2rv\cos(\theta)\cos(\phi) + v^2\cos^2(\phi) + r^2\sin^2(\theta) - 2rv\sin(\theta)\sin(\phi) + v^2\sin^2(\phi)$$
$$= r^2 + v^2 - 2rv\left(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)\right).$$

Then the trigonometric identity hinted at in the ingress gives

$$\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) = \cos(\theta - \phi).$$

Thus we have found

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(r^2+v^2\right)} e^{\frac{rv}{\sigma^2}\cos(\theta-\phi)}.$$

Hence

$$f_R(r) = \int_{-\pi/2}^{\pi/2} f_{R,\Theta}(r,\theta)d\theta =$$
$$= \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(r^2+v^2\right)} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{rv}{\sigma^2}\cos(\theta-\phi)} d\theta.$$

Since $e^{x\cos(\theta)}$ is a periodic function with the period $= 2\pi$, its integrals over any intervals with the length $= 2\pi$ have the same value, we get

$$\int_{-\pi}^{\pi} e^{\frac{rv}{\sigma^2}\cos(\theta-\phi)} d\theta = \int_{-\pi}^{\pi} e^{\frac{rv}{\sigma^2}\cos(\theta)} d\theta.$$

By the theory of Bessel functions we can designate

$$I_0\left(\frac{rv}{\sigma^2}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{rv}{\sigma^2}\cos(\theta)} d\theta.$$

Thus we have shown

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(r^2+v^2\right)} \cdot I_0\left(\frac{rv}{\sigma^2}\right).$$

With $v = 0$ we get here the well known **Rayleigh distribution** $\mathrm{Ra}\left(\sigma^2\right)$ (see the formulas in GUT's Appendix B).

Uppgift 2

We are going to find the characteristic function $\varphi_S(t)$ using the formula

$$\varphi_S(t) = g_N\left(\varphi_X(t)\right),$$

where $g_N(t)$ is the probability generating function of N and $\varphi_X(t)$ is the common characteristic function of the X_is.
By definition we have

$$\varphi_X(t) = E\left[e^{itX}\right] = \sum_{k=1}^{\infty} e^{itk} p_X(k) = \frac{-1}{\ln(1-p)} \sum_{k=1}^{\infty} e^{itk} \frac{p^k}{k} = \frac{-1}{\ln(1-p)} \sum_{k=1}^{\infty} \frac{\left(e^{it}p\right)^k}{k}$$

$$= \frac{1}{\ln(1-p)} \ln\left(1 - e^{-it}p\right),$$

where we used the series expansion for $-\ln(1-x)$ in section 13.3 of *Formulas for probability theory.*
In section 8.1.4 of *Formulas for probability theory* we get that

$$g_N(t) = e^{m(t-1)}$$

where $m = -n \cdot \ln(1-p)$. Thus we have

$$\varphi_S(t) = g_N\left(\varphi_X(t)\right) = e^{m\left(\frac{1}{\ln(1-p)}\ln\left(1-e^{-it}p\right)-1\right)}.$$

Here

$$m\left(\frac{1}{\ln(1-p)} \ln\left(1 - e^{it}p\right) - 1\right) = -n \ln\left(1 - e^{it}p\right) + n \cdot \ln(1-p) =$$

$$= n\left(\ln(1-p) - \ln\left(1 - e^{it}p\right)\right) = n\left(\ln \frac{1-p}{1-e^{it}p}\right).$$

Hence

$$g_N(t) = e^{n \ln\left(\frac{1-p}{1-e^{it}p}\right)} = \left(\frac{1-p}{1-e^{it}p}\right)^n.$$

A glance at the appropriate entry of the formulas in GUT's Appendix B shows that this coincides with the characteristic function of $\mathrm{NBin}(1, 1-p)$. By uniqueness of characteristic functions the claim has been shown.
Comment: The common distribution of the X_is is known as the the *logarithmic series distribution.* Sir Ronald A. Fisher introduced the logarithmic series distribution in ecology as a model of relative species abundance.

Uppgift 3

We expand $\mathrm{Cov}\left(X, g(Y)\right)$ by the pertinent definition and by double expectation.

$$\mathrm{Cov}\left(X, g(Y)\right) = E\left[\left(X - \mu_X\right)\left(g(Y) - E\left[g(Y)\right]\right)\right]$$

By double expectation, as given in section 4 of *Formulas for probability theory*,

$$E\left[(X-\mu_X)\left(g(Y)-E\left[g(Y)\right)\right)\right] = E\left[E\left[(X-\mu_X)\left(g(Y)-E\left[g(Y)\right]\right) \mid Y\right]\right]$$

$$= E\left[\left(g(Y)-E\left[g(Y)\right]\right)E\left[(X-\mu_X) \mid Y\right]\right], \tag{1}$$

where we evoked the rule of taking out what is known, as given in section 4 of *Formulas for probability theory*. We find again in section 9.2 of *Formulas for probability theory* that the conditional distribution of X given Y is

$$N\left(\mu_X + \rho\cdot\frac{\sigma_1}{\sigma_2}(Y-\mu_Y), \sigma_1^2(1-\rho^2)\right).$$

Thus

$$E\left[(X-\mu_X) \mid Y\right] = \rho\cdot\frac{\sigma_1}{\sigma_2}(Y-\mu_Y).$$

When we insert this in the right hand side of (1) we get

$$E\left[\left(g(Y)-E\left[g(Y)\right]\right)E\left[(X-\mu_X) \mid Y\right]\right] = E\left[\left(g(Y)-E\left[g(Y)\right]\right)\rho\cdot\frac{\sigma_1}{\sigma_2}(Y-\mu_Y)\right] =$$

$$= \rho\cdot\frac{\sigma_1}{\sigma_2}E\left[\left(g(Y)-E\left[g(Y)\right]\right)(Y-\mu_Y)\right]$$

$$= \rho\cdot\frac{\sigma_1}{\sigma_2}\mathrm{Cov}\left(Y, g(Y)\right)$$

by definition of covariance. Next, by definition of the coefficient of correlation ρ in section 2.6 of *Formulas for probability theory* we have

$$\rho\cdot\frac{\sigma_1}{\sigma_2} = \frac{\mathrm{Cov}\left(X,Y\right))}{\sigma_1\sigma_2}\cdot\frac{\sigma_1}{\sigma_2} = \frac{\mathrm{Cov}\left(X,Y\right))}{\sigma_2^2}.$$

Therefore we have established the asserted formula as required.

Uppgift 4

If $\{W(t) \mid t \geq 0\}$ is a Wiener process, the Wiener integral

$$X(t) = \int_0^t s dW(s)$$

defines a Gaussian random variable. Also, for any function $\tau(t)$ the random variable

$$W(\tau(t))$$

is Gaussian. Thus, if

$$X(t) \stackrel{d}{=} W(\tau(t))$$

is to hold, $X(t)$ and $W(t)$ must have the same mean and variance. We have by section 10.4 of *Formulas for probability theory* that

$$X(t) \in N\left(0, \int_0^t s^2 ds\right)$$

and by section 10.3 of *Formulas for probability theory*

$$W(\tau(t)) \in N\left(0, \tau(t)\right),$$

where $\tau(t)$ increases monotonically. Thus we must have

$$\tau(t) = \int_0^t s^2 ds \Leftrightarrow \tau(t) = \frac{t^3}{3}.$$

ANSWER: $\underline{\tau(t) = \frac{t^3}{3}, \quad 0 \leq t < 1.}$

Uppgift 5

We set

$$W_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} kX_k.$$

so that

$$S_n = W_n + f(n).$$

We write $W_n = n \cdot X_n + \sum_{k=1}^{n-1} kX_k$. We substitute this expression and use the fact that the independent condition drops out, since X_n is independent of $X_0, \ldots, X_{n-1}$, to get

$$E\left[W_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = E\left[\left(n \cdot X_n + \sum_{k=1}^{n-1} kX_k\right) | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right]$$

and we use linearity

$$= n \cdot E\left[X_n\right] + \sum_{k=1}^{n-1} kE\left[X_k | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right],$$

and we can take out what is known so that $E\left[X_k | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = X_k$, for $k = 1, \ldots, n-1$. Since $X_n \in \text{Po}\left(\frac{1}{n}\right)$ we get

$$= 1 + \sum_{k=1}^{n-1} kX_k = 1 + W_{n-1}.$$

Thus we have found

$$E\left[W_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = 1 + W_{n-1}. \tag{2}$$

Then

$$\begin{aligned} E\left[S_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] &= E\left[W_n + f(n) | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] \\ &= E\left[W_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] + f(n), \end{aligned}$$

and by (2)

$$= 1 + W_{n-1} + f(n),$$

or

$$E\left[S_n | \sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = 1 + W_{n-1} + f(n).$$

Hence, if the martingale property

$$E\left[S_n|\sigma\left(X_0, X_1, \ldots, X_{n-1}\right)\right] = S_{n-1} = W_{n-1} + f(n-1)$$

is to hold, it must be that

$$1 + W_{n-1} + f(n) = W_{n-1} + f(n-1)$$

for integers $n \geq 1$. We get the recursion

$$f(n) = f(n-1) - 1.$$

Since $0 = S_0 = W_0 + f(0) = 0 + f(0) = f(0)$, we get $f(1) = -1$, $f(2) = -2$ e.t.c., which suggests

$$f(n) = -n, \quad n \geq 0,$$

which we can check by an insertion or by an induction. Thus we have established:

$$\text{ANSWER: } \underline{S_n = \sum_{k=1}^{n} k \cdot X_k - n, S_0 = 0.}$$

Comment: When $X_k \in \text{Po}\left(\frac{1}{k}\right)$, we have $E\left[X_k\right] = \frac{1}{k}$. This gives

$$E\left[W_n\right] = \sum_{k=1}^{n} k \cdot E\left[X_k\right] = \sum_{k=1}^{n} k \cdot \frac{1}{k} = \sum_{k=1}^{n} 1 = n. \tag{3}$$

Observe that this holds even for $n = 0$. Hence we have chosen $f(n) = -E\left[W_n\right]$. Note that a martingale must have a constant expectation.

Uppgift 6

If the random variable X has the density

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-\frac{1}{2}(x-a)} & \text{if } x \geq a \\ 0 & \text{if } x < a, \end{cases}$$

where $a > 0$, we get

$$E[X] = \int_a^\infty x f_X(x)\, dx = \int_a^\infty x \frac{1}{2} e^{-\frac{1}{2}(x-a)} dx.$$

We make the change of variable $y = x - a$ to get

$$= \frac{1}{2} \int_0^\infty (y + a)\, e^{-\frac{1}{2}y} dy = \frac{1}{2} \int_0^\infty y e^{-\frac{1}{2}y} dy + a \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}y} dy = 2 + a,$$

since $\frac{1}{2} e^{-\frac{1}{2}y}$ is the density of $\text{Exp}(2)$. In other words, for any i,

$$X_i \stackrel{d}{=} Y + a, \quad Y \in \text{Exp}(2).$$

Therefore, when $S_n = \sum_{i=1}^{n} X_i$ we get

$$E\left[S_n\right] = n\,(2 + a).$$

As the variables are $X_i \stackrel{d}{=} Y + a$, we get also

$$\mathrm{Var}\left[X_i\right] = 4.$$

Then we have the identities

$$\frac{S_n - n(a+2)}{\sqrt{S_n}} = \frac{\frac{1}{\sqrt{n}}\left(S_n - n(a+2)\right)}{\frac{1}{\sqrt{n}}\sqrt{S_n}} =$$

$$\frac{2\frac{1}{2\sqrt{n}}\left(S_n - n(a+2)\right)}{\sqrt{\frac{1}{n}S_n}}.$$

Since the variables are I.I.D., the **central limit theorem** gives

$$\lim_{n\to\infty} \frac{1}{2\sqrt{n}}\left(S_n - n(a+2)\right) \xrightarrow{d} N(0,1).$$

Since the variables are I.I.D., the **law of large numbers** gives

$$\lim_{n\to\infty} \frac{1}{n} S_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = 2 + a$$

in probability. Because $g(x) = \sqrt{x}$ is a continuous function, we have

$$\sqrt{\frac{1}{n}S_n} \xrightarrow{p} \sqrt{2+a},$$

as $n \to \infty$. Therefore the **Cramér-Slutzky theorem** entails

$$\frac{S_n - n(a+2)}{\sqrt{S_n}} \xrightarrow{d} \frac{2}{\sqrt{2+a}} \cdot X,$$

where $X \in N(0,1)$, as $n \to \infty$. The parameter m^2, the variance of $\frac{2}{\sqrt{2+a}} \cdot X$, is thus $\frac{4}{2+a}$.

ANSWER: $\underline{m^2 = \frac{4}{2+a}.}$