Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY SATURDAY THE 22^{nd} OF OCTOBER 2011 9.00 a.m.–2.00 p.m.

Examinator: Timo Koski, tel. 790 71 34, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Saturday 22^{nd} of October 2011 at 4.30 p.m..

The exam results will be announced at the latest on Friday the 4^{th} of November 2011.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

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Uppgift 1

Let $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, $T_3 = \tau_1 + \tau_2 + \tau_3$ be the times of occurrence of the first, second and third event, respectively, in a Poisson process with the intensity $\lambda = 1$.

a) Show that the random vector $(T_1/T_3, T_2/T_3, T_3)$ has the probability density function

$$f_{T_1/T_3, T_2/T_3, T_3}(y_1, y_2, y_3) = y_3^2\, e^{-y_3}, \quad 0 < y_1 < y_2 < 1,\ y_3 > 0.$$

(5 p)

b) Show that $(T_1/T_3, T_2/T_3)$ and T_3 are independent and find their probability density functions. (5 p)

Uppgift 2

X and Y are independent and identically distributed random variables taking values in the positive integers. The probability mass function of X and of Y is

$$p_X(l) = p_Y(l) = p(l) = 2^{-l}, \quad l = 1, 2, 3, \ldots$$

a) Show that $P(Y = X) = \frac{1}{3}$. (4 p)

b) Let k be any positive integer. Show that $P(Y \geq kX) = \frac{2}{2^{k+1}-1}$. (5 p)

c) Find $P(Y > X)$. (1 p)

Uppgift 3

Let $S = X_1 + X_2 + \cdots + X_r$, where $X_1, X_2, \ldots$ are independent and identically distributed with the probability mass function $p(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots;\ 0 < p < 1$.

a) What is the distribution of S ?
Give an interpretation of your result in words, too. (4 p)

b) Show that pS converges in distribution to $\Gamma(r, 1)$, as $p \to 0$. (6 p)

Uppgift 4

$X_1, X_2, \ldots$ are independent and identically distributed random variables with $X_i \in \mathrm{Po}(1)$ for $i = 1, 2, \ldots,$. The random variable N is independent of the X_i's, and $N \in \mathrm{Po}(2)$. We set

$$S_N = X_1 + X_2 + \cdots + X_N, \quad S_0 = 0.$$

Find the probability

$$P(S_N = 0).$$

(10 p)

Uppgift 5

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process and the process $\mathbf{X} = \{X(t) \mid t \geq 0\}$ is with $\lambda > 0$ defined by

$$X(t) = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-u)} dW(u), \quad t \geq 0.$$

Here X_0 is a random variable independent of $\mathbf{W}$. Let $\mathcal{F}_t^{W,X_0}$ be the sigma-field generated by X_0 and the process $\mathbf{W}$ up to time t, .i.e., $\mathcal{F}_t^{W,X_0} = \sigma\left(X_0, W(u) \mid 0 \leq u \leq t\right)$.

a) Show that for $t > s > 0$

$$E\left[X(t) \mid \mathcal{F}_s^{W,X_0}\right] = e^{-\lambda(t-s)} X(s).$$

Be so kind and justify Your steps of solution carefully. (5 p)

b) Assume now in addition that $X_0 \in N\left(0, \frac{1}{2\lambda}\right)$. Then with $h = t - s$ we have

$$R_{\mathbf{X}}(h) = E\left[X(t)X(s)\right] = \frac{1}{2\lambda} e^{-\lambda|h|}.$$

You need not check or show this. Find a number β such that for $t > s > 0$ the probability

$$P\left(\mid X(t) - \beta X(s) \mid \geq k\right)$$

is minimized for every $k > 0$. (5 p)

Uppgift 6

The process $\mathbf{X} = \{X(t) \mid -\infty < t < \infty\}$ is Gaussian and weakly stationary. Its mean function is $= 0$ and its autocorrelation function is

$$R_X(h) = \frac{2}{\pi} \cdot \frac{\sin(h)}{h}, \quad -\infty < h < \infty.$$

We sample this process in time so that our samples are

$$X_k = X(\pi k), \quad k = 1, 2, \ldots.$$

Show that

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k^2}} \to N\left(0, \sigma^2\right),$$

as $n \to +\infty$ and determine the variance σ^2. Be so kind and justify Your steps of solution very carefully. (10 p)

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SOLUTIONS TO THE EXAM SATURDAY THE 22^{nd} OF JANUARY 2011 09.00 a.m.–02.00 p.m..

Uppgift 1

a) For a Poisson process with the intensity 1 we have that τ_1, τ_2, τ_3 are independent with distribution Exp(1) and have thus the density e^{-x}, $x > 0$ and 0 elsewhere. The invertible transformation

$$y_1 = \tau_1/(\tau_1 + \tau_2 + \tau_3),$$

$$y_2 = (\tau_1 + \tau_2)/(\tau_1 + \tau_2 + \tau_3),$$

$$y_3 = \tau_1 + \tau_2 + \tau_3$$

maps the domain $\{\tau_1 > 0, \tau_2 > 0, \tau_3 > 0\}$ on $\{0 < y_1 < y_2 < 1, y_3 > 0\}$. The inverse is given by

$$\tau_1 = y_1 y_3 = h_1(y_1, y_2, y_3),$$

$$\tau_2 = (y_2 - y_1)y_3 = h_2(y_1, y_2, y_3),$$

$$\tau_3 = (1 - y_2)y_3 = h_3(y_1, y_2, y_3).$$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial \tau_1}{\partial y_1} & \frac{\partial \tau_1}{\partial y_2} & \frac{\partial \tau_1}{\partial y_3} \\ \frac{\partial \tau_2}{\partial y_1} & \frac{\partial \tau_2}{\partial y_2} & \frac{\partial \tau_2}{\partial y_3} \\ \frac{\partial \tau_3}{\partial y_1} & \frac{\partial \tau_3}{\partial y_2} & \frac{\partial \tau_3}{\partial y_m} \end{vmatrix} = \begin{vmatrix} y_3 & 0 & y_1 \\ -y_3 & y_3 & y_2 - y_1 \\ 0 & -y_3 & 1 - y_2 \end{vmatrix}$$

$$= y_3\left(y_3(1-y_2) + y_3(y_2 - y_1)\right) + y_1 y_3^2 = y_3^2.$$

The theorem about transformation of variables gives

$$f_{T_1/T_3, T_2/T_3, T_3}(y_1, y_2, y_3) = f_{\tau_1,\tau_2,\tau_3}(h_1(y_1,y_2,y_3), h_2(y_1,y_2,y_3), h_3(y_1,y_2,y_3))|J|$$

$$= e^{-y_1 y_3} e^{-(y_2 - y_1)y_3} e^{-(1-y_2)y_3} |J|$$

$$= e^{-y_3}|J| = e^{-y_3}\, y_3^2,$$

as was to be shown.

b) We write $e^{-y_3}\, y_3^2 = \frac{y_3^{(3-1)}}{\Gamma(3)} e^{-y_3}\Gamma(3)$, and observe by Gut Appendix 2 that $f_{T_3}(y_3) = \frac{y_3^{(3-1)}}{\Gamma(3)} e^{-y_3}$ is the density of $\Gamma(3,1)$. Because $\Gamma(3) = 2$, we have

$$f_{T_1/T_3, T_2/T_3, T_3}(y_1, y_2, y_3) = 2 \cdot f_{T_3}(y_3), \quad 0 < y_1 < y_2 < 1,\ y_3 > 0.$$

It is follows by a straightforward integration that

$$\int_0^1 \int_0^{y_2} 2 dy_1 dy_2 = 1.$$

Hence $f_{T_1/T_3,T_2/T_3}(y_1,y_2) = 2$ on $0 < y_1 < y_2 < 1$ and

$$f_{T_1/T_3,T_2/T_3,T_3}(y_1,y_2,y_3) = f_{T_1/T_3,T_2/T_3}(y_1,y_2) \cdot f_{T_3}(y_3).$$

and the claim follows as asserted.

Uppgift 2

a) Since X and Y are independent and identically distributed random variables we get

$$P(Y = X) = \sum_{l=1}^{\infty} P(Y = l, X = l) = \sum_{l=1}^{\infty} P(Y = l)\, P(X = l) = \sum_{l=1}^{\infty} 2^{-l} \cdot 2^{-l} = \sum_{l=1}^{\infty} 2^{-2l}$$

and by geometric series

$$= \sum_{l=0}^{\infty} 2^{-2l} - 1 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{4}{3} - 1 = \frac{1}{3},$$

as was claimed.

b) k is now any positive integer.

$$P(Y \geq kX) = \sum_{l=1}^{\infty} P(Y \geq kX, X = l)$$

and by independence

$$= \sum_{l=1}^{\infty} P(Y \geq kl, X = l) = \sum_{l=1}^{\infty} P(Y \geq kl)\, P(X = l).$$

We write

$$P(Y \geq kl) = \sum_{j=kl}^{\infty} 2^{-j} = \sum_{y=0}^{\infty} 2^{-y-kl},$$

where we made the change of the variable of summation $y = j - kl \Leftrightarrow j = y + lk$. Then

$$\sum_{l=1}^{\infty} P(Y \geq kl)\, P(X = l) = \sum_{l=1}^{\infty}\sum_{y=0}^{\infty} 2^{-y-kl}2^{-l}$$

$$= \sum_{l=1}^{\infty} 2^{-kl}2^{-l} \sum_{y=0}^{\infty} 2^{-y} = \sum_{l=1}^{\infty} 2^{-kl}2^{-l} \frac{1}{1-1/2} =$$

$$= 2\sum_{l=1}^{\infty} 2^{-(k+1)l} = 2\left(\sum_{l=0}^{\infty} 2^{-(k+1)l} - 1\right)$$

$$= 2\left(\frac{1}{1-1/2^{k+1}} - 1\right) = 2\left(\frac{2^{k+1} - \left(2^{k+1}-1\right)}{2^{k+1}-1}\right) = \frac{2}{2^{k+1}-1}.$$

This establishes the claim as asserted.

c) By basic rules of probability calculus we have

$$P(Y > X) = P(Y \geq X) - P(Y = X).$$

By part a) and by part b) (with $k = 1$) above we get

$$P(Y > X) = \frac{2}{2^2 - 1} - \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

ANSWER (c): $\underline{P(Y > X) = \frac{1}{3}}$.

Uppgift 3

a) A random variable X with the probability mass function $p(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$ is by Gut Appendix 2 $Fs(p)$- distributed. Hence the characteristic function is with $q = 1 - p$

$$\varphi_X(t) = \frac{pe^{it}}{1 - qe^{it}}.$$

Therefore the sum $S = X_1 + X_2 + \cdots + X_r$ has the characteristic function

$$\varphi_{S_r}(t) = (\varphi_X(t))^r = \left(\frac{pe^{it}}{1 - qe^{it}}\right)^r.$$

We write this as

$$\varphi_{S_r}(t) = e^{irt}\left(\frac{p}{1 - qe^{it}}\right)^r = e^{irt}\varphi_Y(t),$$

where

$$\varphi_Y(t) = \left(\frac{p}{1 - qe^{it}}\right)^r$$

is the characteristic function of the negative binomial distribution $\mathrm{NBin}(r, p)$ as found in Gut Appendix B. Thus by the translation rule of characteristic functions and by uniqueness of characteristic functions

$$S_r \stackrel{d}{=} Y + r, \quad Y \in \mathrm{NBin}(r, p).$$

Therefore it holds for any integer $k > 0$

$$P(S_r = k) = P(Y + r = k) = P(Y = k - r).$$

By properties of negative binomial distribution we note that

$$P(Y = k - r) = 0, \quad \text{if } k < r$$

and

$$P(Y = k - r) = \binom{r + (k - r) - 1}{k - r} p^r q^{k-r}, \quad k = r, r + 1, \ldots$$

I.e., the answer is

$$P(S_r = k) = \begin{cases} 0, & \text{if } k < r \\ \binom{k-1}{k-r} p^r q^{k-r}, & k = r, r+1, \ldots \end{cases}.$$

It is easily checked that

$$\binom{k-1}{k-r} = \binom{k-1}{r-1},$$

so that

$$P(S_r = k) = \begin{cases} 0, & \text{if } k < r \\ \binom{k-1}{r-1} p^r q^{k-r}, & k = r, r+1, \ldots \end{cases}.$$

This last expression is occasionally known as the probability mass function of the *Pascal distribution*.

> There is a fair deal of confusing variation w.r.t. the terminology. Sometimes the Pascal distribution defined as above is called the negative binomial distribution. In some textbooks the negative binomial distribution as defined in Gut is known as the Pascal distibution.

Let us try to explain in words. A random variable Y with the negative binomial distribution $\mathrm{NBin}(r, p)$ is (in distribution) the sum of r independent $Z_i \in \mathrm{Ge}(p)$ and it holds that

$$X_i = Z_i + 1, \quad i = 1, 2, \ldots, r$$

as any Z_i is the number of independent binary trials (trials with two outcomes) before you get the first success ($p =$ probability of success in a single trial), and X_i is the the number of trials before you get the first success plus the successful trial. Then

$$S_r = X_1 + X_2 + \cdots + X_r = Z_1 + Z_2 + \cdots + Z_r + r = Y + r.$$

$S_r =$ the number of independent trials you need to do in order to get r successes plus the rth success.

b) Part a) gives that the characteristic function of pX is

$$E(e^{itpX}) = (pe^{itp}/(1 - (1-p)e^{itp}))^r$$

A Taylorian epxansion yields

$$= (p(1 + O(p))/(1 - (1-p)(1 + itp + O(p^2))))^r = (1/(1 - it + O(p)))^r \to 1/(1-it)^r.$$

Since the limit is the characteristic function of $\Gamma(r, 1)$- distribution, the assertion follows by the theorem of convergence for characteristic functions.

Uppgift 4

We use the method of probability generating functions (p.g.f.s). By a well known result, as X_is are independent and identically distributed random variables independent of N that the p.g.f. $g_{S_N}(t)$ of S_N is

$$g_{S_N}(t) = g_N(g_X(t)),$$

where $g_X(t)$ is the p.g.f. of $X \stackrel{d}{=} X_i$. Since $X_i \in \mathrm{Po}(1)$ and $N \in \mathrm{Po}(2)$ we get by the Collection of Formulas section 8.1.4. that

$$g_X(t) = e^{(t-1)}, \quad g_N(t) = e^{2(t-1)}.$$

Thereby we get

$$g_{S_N}(t) = g_N\left(e^{(t-1)}\right) = e^{2(e^{(t-1)}-1)}.$$

Then, by a property of p.g.f.s

$$P\left(S_N = 0\right) = g_{S_N}(0) = e^{2(e^{-1}-1)}.$$

$$\text{ANSWER: } \underline{P\left(S_N = 0\right) = e^{2(e^{-1}-1)}.}$$

Uppgift 5

a) By construction we get that

$$E\left[X(t) \mid \mathcal{F}_s^{W,X_0}\right] = E\left[e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right]$$

and by a property of Wiener integral

$$= E\left[e^{-\lambda t}X_0 + \int_s^t e^{-\lambda(t-u)}dW(u) + \int_0^s e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right]$$

and by linearity of conditional expectation this equals

$$= E\left[e^{-\lambda t}X_0 \mid \mathcal{F}_s^{W,X_0}\right] + E\left[\int_s^t e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right] + E\left[\int_0^s e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right].$$

When we take out what is known (a rule of conditional expectation in Collection of Formulas) or by measurability, we get in the first term on the right hand side

$$E\left[e^{-\lambda t}X_0 \mid \mathcal{F}_s^{W,X_0}\right] = e^{-\lambda t}X_0,$$

and by the fact that the increments of a Wiener process in $[s,t]$ are independent, and thus independent of $\mathcal{F}_s^{W,X_0}$, a rule for computation with conditional expectations (an independent condition drops out) gives

$$E\left[\int_s^t e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right] = E\left[\int_s^t e^{-\lambda(t-u)}dW(u)\right] = 0,$$

by a property of the Wiener integral. In the third term we are again allowed to take out what is known and get

$$E\left[\int_0^s e^{-\lambda(t-u)}dW(u) \mid \mathcal{F}_s^{W,X_0}\right] = \int_0^s e^{-\lambda(t-u)}dW(u).$$

Thus we have obtained

$$E\left[X(t) \mid \mathcal{F}_s^{W,X_0}\right] = e^{-\lambda t}X_0 + \int_0^s e^{-\lambda(t-u)}dW(u)$$

and some algebra entails

$$= e^{-\lambda(t-s)}e^{-\lambda s}X_0 + e^{-\lambda(t-s)}\int_0^s e^{-\lambda(s-u)}dW(u)$$

$$= e^{-\lambda(t-s)} \underbrace{\left(e^{-\lambda s} X_0 + \int_0^s e^{-\lambda(s-u)} dW(u) \right)}_{=X(s)}$$

i.e., by definition of the process $\mathbf{X}$ this is

$$= e^{-\lambda(t-s)} X(s),$$

as was asserted.

b) We have

$$E\left[X(t) - \beta X(s)\right] = 0,$$

and we set for ease of writing $\sigma^2 = \text{Var}\left[X(t) - \beta X(s)\right]$ and get

$$\sigma^2 = \text{Var}[X(t)] + \beta^2 \text{Var}\left[X(s)\right] - 2\beta R_X(t-s).$$

We have

$$\text{Var}[X(t)] = \text{Var}\left[X(s)\right] = R_{\mathbf{X}}(0) = \frac{1}{2\lambda}$$

and since $0 < s < t$,

$$R_{\mathbf{X}}(t-s) = \frac{1}{2\lambda} e^{-\lambda(t-s)}.$$

Let us set $Z = X(t) - \beta X(s)$. Then

$$Z \in N\left(0, \sigma^2\right).$$

Thus, by probability of the complement event and as Z is a continuous r.v.,

$$P\left(\mid Z \mid \geq k\right) = 1 - P\left(\mid Z \mid \leq k\right) = 1 - P\left(-k \leq Z \leq k\right)$$

$$= 1 - P\left(\frac{-k}{\sigma} \leq \frac{Z}{\sigma} \leq \frac{k}{\sigma} \right)$$

where $\frac{Z}{\sigma} \in N(0,1)$, and thus

$$= 1 - \left(\Phi\left(\frac{k}{\sigma}\right) - \Phi\left(\frac{-k}{\sigma}\right) \right),$$

where $\Phi(x)$ is the distribution function of $N(0,1)$. We have that $\Phi(-x) = 1 - \Phi(x)$ and thus

$$= 1 - \left(\Phi\left(\frac{k}{\sigma}\right) - (1 - \Phi\left(\frac{k}{\sigma}\right) \right)$$

$$= 2\left(1 - \Phi\left(\frac{k}{\sigma}\right)\right).$$

We see that this probability is minimized, as soon as σ is minimised as a function of β. This is equivalent to minimizing σ^2. We differentiate w.r.t. β and

$$\frac{d}{d\beta}\sigma^2 = \frac{d}{d\beta}\left[\frac{1}{2\lambda} + \beta^2 \frac{1}{2\lambda} - 2\beta \frac{1}{2\lambda} e^{-\lambda(t-s)} \right]$$

$$= 2\beta\frac{1}{2\lambda} - 2\frac{1}{2\lambda}e^{-\lambda(t-s)}].$$

We set the derivative $\frac{d}{d\beta}\sigma^2 = 0$ and solve w.r.t. β, which gives

$$\beta = e^{-\lambda(t-s)}.$$

A check of the second derivative shows that this gives the minimum.
ANSWER: $\underline{\beta = e^{-\lambda(t-s)}}$.

Uppgift 6

The sampled variables

$$X_k = X(\pi k), \quad k = 1, 2, \ldots,$$

are Gaussian, and

$$E[X_k] = E[X(\pi k)] = 0 \quad k = 1, 2, \ldots,$$

since the mean function is the constant zero. In addition

$$\text{Var}[X_k] = \text{Var}[X(\pi k)] = R_X(0) = \frac{2}{\pi} \cdot \frac{\sin(\pi 0)}{\pi 0}$$

We use the interpretation of $\frac{\sin(\pi 0)}{\pi 0}$ via the limit

$$\lim_{h\to 0}\frac{\sin(\pi h)}{h} = \lim_{h\to 0}\frac{\sin(\pi h) - 0}{h - 0}$$

$$= \frac{d}{dx}\sin(\pi x)\mid_{x=0} = \pi\cos(0) = \pi.$$

i.e.,

$$\text{Var}[X_k] = \frac{2}{\pi}.$$

Furthermore, for $k \neq r$

$$E[X_k X_r] = R_X(\pi(k-r)) = \frac{2}{\pi} \cdot \frac{\sin(\pi(k-r))}{\pi(k-r)} = 0,$$

since $\sin(\pi(k-r)) = 0$, as $k - r$ is an integer.
In other words, the random variables $X_k, k = 1, 2, \ldots,$ are non-correlated, and thus independent, as they are Gaussian. *The random variables $X_k, k = 1, 2, \ldots,$ are therefore independent and identically distributed (I.I.D.),*

$$X_k \in N(0, a^2), \quad a^2 = \frac{2}{\pi}.$$

Due to this crucial fact (which must be stated in order to get full credit points, 'justify ... very carefully'.) we are allowed to apply the central limit theorem and the law of large numbers as formulated and proved in sf2940, to study

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k^2}},$$

as $n \to \infty$. We write

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k^2}} = \frac{\frac{1}{a\sqrt{n}} \sum_{k=1}^{n} X_k}{\sqrt{\frac{1}{a^2 n} \sum_{k=1}^{n} X_k^2}}.$$

Since the variables are I.I.D., the **central limit theorem** entails

$$\lim_{n\to\infty} \frac{1}{a\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow{d} N(0,1).$$

Of course, $\frac{1}{a\sqrt{n}} \sum_{k=1}^{n} X_k \in N(0,1)$ for any n.
Since the variables are I.I.D., the **law of large numbers** gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k^2 = E\left[X^2\right]$$

in probability, where $E\left[X^2\right] = a^2$. Because $g(x) = \sqrt{x}$ is a continuous function, we have

$$\sqrt{\frac{1}{a^2 n} \sum_{k=1}^{n} X_k^2} \xrightarrow{p} \sqrt{1},$$

as $n \to \infty$. Therefore the **Cramér-Slutzky theorem** entails

$$\frac{\frac{1}{a\sqrt{n}} \sum_{k=1}^{n} X_k}{\sqrt{\frac{1}{an} \sum_{k=1}^{n} X_k^2}} \xrightarrow{d} Z,$$

where $Z \in N(0,1)$, as $n \to \infty$. The parameter σ^2 is thus $= 1$.

ANSWER: $\underline{\sigma^2 = 1.}$