Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY THE 17^{th} OF OCTOBER 2012 8.00 a.m.–01.00 p.m.

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering.* Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 17^{th} of October 2012 at 4.30 p.m..

The exam results will be announced at the latest on Friday the 26^{th} of October 2012.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

$X \in \text{Exp}(1)$, $Y \in \text{Exp}(1)$, and X and Y are indepedent. Set

$$U = X + Y, \quad V = \frac{X}{X+Y}.$$

a) Find the marginal distributions of U and V. Which known two distributions (i.e. distributions found in the appendix B) are these ? (8 p)

b) Show that U and V are independent. Justify your answer. (2 p)

Uppgift 2

Consider the sequence of random variables X_n, $n = 2, 3, \ldots$, such that $X_n \in \text{Bin}\left(n, \frac{n+1}{n^2}\right)$. Show that

$$X_n \xrightarrow{d} X,$$

as $n \to \infty$, and find the probability distribution of X. Justify your solution carefully. (10 p)

Uppgift 3

Let $\mathbf{X} = (X_1, X_2)' \in N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}\right)$. Be kind and compute

$$P\left(X_2 - E\left[X_2 \mid X_1\right] > 0.1\right).$$

(10 p)

Uppgift 4

Let $\{W(t) \mid t \geq 0\}$ be a Wiener process. Define

$$Y(t) = 2tW\left(\frac{1}{4t}\right), \quad t > 0, \quad Y(0) = 0.$$

a) Show that $Y(t) - Y(s) \in N(0, t-s)$ for $0 \leq s < t$. (2 p)

b) Show that the increments $Y(t_i) - Y(t_{i-1})$ are independent random variables for $i = 1, \ldots, n$, where $0 \leq t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t_n$. (8 p)

Uppgift 5

$\mathbf{N} = \{N(t) \mid t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$. Its autocorrelation function is (you need not check or prove this)

$$R_{\mathbf{N}}(t,s) = \lambda^2 t \cdot s + \lambda \min(t,s).$$

the autocovariance function of $\mathbf{N}$ is (you need not check or prove this)

$$\mathrm{Cov}_{\mathbf{N}}(t,s) = R_{\mathbf{N}}(t,s) - \mu_{\mathbf{N}}(t)\mu_{\mathbf{N}}(s) = \lambda \min(t,s).$$

We define the new process $\mathbf{Y} = \{Y(t) \mid 0 \leq t \leq 1\}$ by

$$Y(t) \stackrel{\text{def}}{=} N(t) - tN(1), \quad 0 \leq t \leq 1.$$

a) Are the sample paths of $\mathbf{Y}$ (OBS !: change from 17/10) nondecreasing ? Justify your answer. (1 p)

b) Compute $E[Y(t)]$. Show your calculations. (1 p)

c) Compute $\mathrm{Var}[Y(t)]$. Show your calculations. (2 p)

d) Find the autocovariance function of $\mathbf{Y}$. Show your calculations. (6 p)

Uppgift 6

a) $\{X_n\}_{n\geq 1}$ is a sequence of indepedent r.v.'s with the common characteristic function $\varphi_X(t)$, and N is independent of $\{X_n\}_{n\geq 1}$ and has the probability generating function $g_N(t)$. Set $S_N = X_1 + X_2 + \ldots + X_N$. Show that the characteristic function of S_N is given by the composition formula

$$\varphi_{S_N}(t) = g_N(\varphi_X(t)).$$

(2 p)

b) $\{X_n\}_{n\geq 1}$ is a sequence of indepedent r.v.'s such that

$$\underline{\mathrm{P}}(X_n = x) = \begin{cases} \frac{1}{2} & x = -1 \\ \frac{1}{2} & x = 1. \end{cases}$$

Let $N \in \mathrm{Po}(\lambda)$. N is independent of $\{X_n\}_{n\geq 1}$. Set $S_N = X_1 + X_2 + \ldots + X_N$. Show that the characteristic function of S_N is

$$\varphi_{S_N}(t) = e^{-\lambda(1-\cos(t))}.$$

(3 p)

c) Show that for S_N in (b)

$$\frac{S_N}{\sqrt{\lambda}} \stackrel{d}{\to} N(0,1), \tag{1}$$

as $\lambda \to \infty$. *Aid*: You may need a series expansion from L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering.* (3 p)

d) Explain how (1) is related to the central limit theorem. You may overlook the fact that λ is not an integer. (2 p)

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SOLUTIONS TO THE EXAM SATURDAY THE 17^{th} OF OCTOBER 2012 08.00 a.m.–01.00 p.m..

Uppgift 1

a) We write for $x, y > 0$

$$v = g_1(x,y) = x/(x+y), u = g_2(x,y) = x+y$$

and obtain therefore the inverse transformation

$$x = h_1(u,v) = uv, y = h_2(u,v) = (1-v)u \quad \text{for } 0 < u < 1, v > 0.$$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} = -vu - u(1-v) = -u < 0.$$

The theorem about transformation of variables gives

$$f_{U,V}(u,v) = f_{X,Y}(uv, (1-v)u)|J|$$

and since $X \in \text{Exp}(1)$ and $Y \in \text{Exp}(1)$ are independent, and by the expression for the Jacobian, the rule for change of variable in a probability density gives

$$= e^{-uv}e^{-(1-v)u}uI(1 > u > 0, v > 0) = ue^{-u}I(1 > v > 0, u > 0).$$

where $I(1 > v > 0, u > 0)$ is the indicator function of the domain.

$$\text{ANSWER a): } \underline{f_{U,V}(u,v) = ue^{-u}I(1 > v > 0, u > 0).}$$

b) If U and V are to be independent, their joint density must be factorized for all (u,v). We observe that a property of indicator functions gives

$$I(1 > v > 0, u > 0) = I(1 > v > 0)I(u > 0)$$

Thus we conclude from a) that

$$f_{U,V}(u,v) = I(1 > v > 0) \cdot ue^{-u}I(u > 0).$$

Hence we can write $f_{U,V}(u,v) = f_U(u)f_V(v)$ for all (u,v), where

$$f_V(v) = I(0 < v < 1), \quad f_U(u) = ue^{-u}I(u > 0).$$

Here we recognize $f_U(u)$ as the probability density of $\Gamma(2,1)$ and $f_V(v)$ as the probability density of $U(0,1)$.

ANSWER b): $\underline{U \in U(0,1), V \in \Gamma(2,1), U \text{ and } V \text{ are independent.}}$

Uppgift 2

If $X_n \in \text{Bin}\left(n, \frac{n+1}{n^2}\right)$, then its characteristic function is from Appendix B with $q = 1 - \frac{n+1}{n^2}$, $p = \frac{n+1}{n^2}$, and $n \geq 2$

$$\varphi_{X_n}(t) = \left(q + pe^{it}\right)^n = \left(1 - \frac{n+1}{n^2} + \frac{n+1}{n^2}e^{it}\right)^n$$

$$= \left(1 + \frac{1}{n}\left(-\frac{n+1}{n} + \frac{1}{n}e^{it} + e^{it}\right)\right)^n$$

$$= \left(1 + \frac{1}{n}\left(e^{it} - 1 - \frac{1}{n} + \frac{1}{n}e^{it}\right)\right)^n$$

We set

$$c_n \stackrel{\text{def}}{=} \left(e^{it} - 1 - \frac{1}{n} + \frac{1}{n}e^{it}\right).$$

Then

$$c_n \to c = e^{it} - 1,$$

as $n \to \infty$. Therefore, section 13.3. in the Collection of Formulas gives,

$$\left(1 + \frac{c_n}{n}\right)^n \to e^c = e^{e^{it}-1},$$

as $n \to \infty$. By Appendix B, $e^{e^{it}-1}$ is the characteristic function of Po(1). $e^{e^{it}-1}$ is continuous at $t = 0$. Hence the continuity theorem for characteristic functions yields that

$$X_n \stackrel{d}{\to} X,$$

as $n \to \infty$, where $X \in \text{Po}(1)$. (10 p)

ANSWER : $\underline{X_n \stackrel{d}{\to} \text{Po}(1).}$

Uppgift 3

The Collection of Formulas section 9.2. gives

$$E\left[X_2 \mid X_1\right] = \mu_{X_2} + \rho\frac{\sigma_{X_2}}{\sigma_{X_1}}\left(X_1 - \mu_{X_1}\right), \tag{2}$$

and amongst other things this tells that $E\left[X_2 \mid X_1\right]$ is a random variable with a normal distribution. In this case

$$\mu_{X_2} = 2, \mu_{X_1} = 1, \rho = 0.6, \sigma_{X_2} = \sigma_{X_1} = 1.$$

We set

$$Y \stackrel{\text{def}}{=} X_2 - E\left[X_2 \mid X_1\right],$$

and thus Y is a r.v. with a normal distribution. We need its mean and variance. Then

$$E\left[Y\right] = E\left[X_2\right] - E\left[E\left[X_2 \mid X_1\right]\right] = E\left[X_2\right] - E\left[X_2\right] = 0,$$

by double expectation, or directly by (2).

$$Y = X_2 - E\left[X_2 \mid X_1\right] = X_2 - \mu_{X_2} - \rho\left(X_1 - \mu_{X_1}\right),$$

and

$$\begin{aligned}
\text{Var}\left[Y\right] &= \text{Var}\left[X_2 - \mu_{X_2}\right] + \text{Var}\left[\rho\left(X_1 - \mu_{X_1}\right)\right] - 2\text{Cov}(X_2 - \mu_{X_2}, \rho\left(X_1 - \mu_{X_1}\right)). \\
&= 1 + \rho^2 - 2\rho\text{Cov}\left(\left(X_2 - \mu_{X_2}\right), \left(X_1 - \mu_{X_1}\right)\right) \\
&= 1 + \rho^2 - 2\rho\left(E\left[\left(\left(X_2 - \mu_{X_2}\right) \cdot \left(X_1 - \mu_{X_1}\right)\right] - E\left[\left(X_2 - \mu_{X_2}\right)\right) E\left[\left(\left(X_1 - \mu_{X_1}\right)\right]\right)\right. \\
&= 1 + \rho^2 - 2\rho E\left[\left(X_2 - \mu_{X_2}\right)\left(X_1 - \mu_{X_1}\right)\right] - \\
&= 1 + \rho^2 - 2\rho\text{Cov}\left(X_2, X_1\right)) \\
&= 1 + \rho^2 - 2\rho^2 = 1 - \rho^2 = 1 - 0.6^2.
\end{aligned}$$

Alternatively

$$\text{Var}\left[Y\right] = \text{Var}\left[X_2\right] + \text{Var}\left[E\left[X_2 \mid X_1\right]\right] - 2\text{Cov}(X_2, E\left[X_2 \mid X_1\right]). \tag{3}$$

Here by definition

$$\text{Cov}(X_2, E\left[X_2 \mid X_1\right]) = E\left[X_2 \cdot E\left[X_2 \mid X_1\right]\right] - E\left[X_2\right] \cdot E\left[E\left[X_2 \mid X_1\right]\right].$$

In the second term double expectation gives again

$$E\left[E\left[X_2 \mid X_1\right]\right] = E\left[X_2\right].$$

Hence

$$\text{Cov}(X_2, E\left[X_2 \mid X_1\right]) = E\left[X_2 \cdot E\left[X_2 \mid X_1\right]\right] - \left(E\left[X_2\right]\right)^2.$$

The first term is evaluated by double expectation

$$E\left[X_2 \cdot E\left[X_2 \mid X_1\right]\right] = E\left[E\left[X_2 \cdot E\left[X_2 \mid X_1\right] \mid X_1\right]\right] = E\left[E\left[X_2 \mid X_1\right] \cdot E\left[X_2 \mid X_1\right]\right] =$$

where we took out what is known (rule 3. in section 4 of the Collection of Formulas), as $E\left[X_2 \mid X_1\right]$ is measurable w.r.t. the sigma field generated by X_1. Then

$$E\left[E\left[X_2 \mid X_1\right] \cdot E\left[X_2 \mid X_1\right]\right] = E\left[\left(E\left[X_2 \mid X_1\right]\right)^2\right].$$

Thus

$$\text{Cov}(X_2, E\left[X_2 \mid X_1\right]) = E\left[\left(E\left[X_2 \mid X_1\right]\right)^2\right] - \left(E\left[X_2\right]\right)^2.$$

Since $E\left[E\left[X_2 \mid X_1\right]\right] = E\left[X_2\right]$, as stated above, we get

$$\text{Cov}(X_2, E\left[X_2 \mid X_1\right]) = \text{Var}\left[E\left[X_2 \mid X_1\right]\right].$$

When this is inserted in (3) we get

$$\mathrm{Var}\left[Y\right] = \mathrm{Var}\left[X_2\right] + \mathrm{Var}\left[E\left[X_2 \mid X_1\right]\right] - 2\mathrm{Var}\left[E\left[X_2 \mid X_1\right]\right],$$

or,

$$\mathrm{Var}\left[Y\right] = \mathrm{Var}\left[X_2\right] - \mathrm{Var}\left[E\left[X_2 \mid X_1\right]\right].$$

Since from (2),

$$E\left[X_2 \mid X_1\right] = \mu_{X_2} + \rho\frac{\sigma_{X_2}}{\sigma_{X_1}}\left(X_1 - \mu_{X_1}\right),$$

we get by the specifics in the Uppgift that

$$\mathrm{Var}\left[E\left[X_2 \mid X_1\right]\right] = \rho^2\mathrm{Var}\left[X_1\right] = 0.6^2 \cdot 1 = 0.36.$$

and

$$\mathrm{Var}\left[Y\right] = 1 - \rho^2 = 1 - 0.36 = 0.64.$$

Thus

$$Y \in N\left(0, 0.64\right).$$

Hence $\frac{Y}{\sqrt{0.64}} \in N(0,1)$ and the desired probability is

$$P\left(Y > 0.1\right) = P\left(\frac{Y}{\sqrt{0.64}} > \frac{0.1}{\sqrt{0.64}}\right) =$$

$$= 1 - \Phi\left(\frac{0.1}{\sqrt{0.64}}\right) \approx 1 - \Phi\left(0.125\right) \approx 0.453.$$

$$\text{ANSWER}: \underline{P\left(X_2 - E\left[X_2 \mid X_1\right] > 0.1\right) = 1 - \Phi\left(\tfrac{0.1}{\sqrt{0.64}}\right).}$$

Uppgift 4

a) Since $\{W(t) \mid t \geq 0\}$ is a Wienerprocess, then it is a Gaussian process. When $t > s > 0$, $Y(t) - Y(s)$ is a Gaussian r.v. and

$$E\left[Y(t) - Y(s)\right] = E\left[2tW\left(\frac{1}{4t}\right)\right] - E\left[2sW\left(\frac{1}{4s}\right)\right] = 0 - 0 = 0,$$

since the mean of the Wienerprocess is $= 0$. We need the variance of $Y(t) - Y(s)$.

$$\mathrm{Var}\left[Y(t) - Y(s)\right] = \mathrm{Var}\left[2tW\left(\frac{1}{4t}\right)\right] + \mathrm{Var}\left[2sW\left(\frac{1}{4s}\right)\right] - 2\mathrm{Cov}\left(2tW\left(\frac{1}{4t}\right), 2sW\left(\frac{1}{4s}\right)\right)$$

$$= 4t^2\frac{1}{4t} + 4s^2\frac{1}{4s} - 2E\left[2tW\left(\frac{1}{4t}\right) \cdot 2sW\left(\frac{1}{4s}\right)\right]$$

since $W(t) \in N(0,t)$. Since $E\left[W(t) \cdot W(s)\right] = \min(t,s)$, we get,as $t > s$ (so that $\frac{1}{4t} < \frac{1}{4s}$)

$$2E\left[2tW\left(\frac{1}{4t}\right) \cdot 2sW\left(\frac{1}{4s}\right)\right] = 8ts\min\left(\frac{1}{4t}, \frac{1}{4s}\right) = 8ts\frac{1}{4t} = 2s.$$

Hence

$$\text{Var}\left[Y(t) - Y(s)\right] = t + s - 2s = t - s.$$

b) We know that the process $\{Y(t) \mid 0 \le t\}$ is a Gaussian process, since $\{W(t) \mid t \ge 0\}$ is a Wienerprocess. We have in a) shown that

$$E\left[Y(t)\right] = 0.$$

In addition, if $t > s$, as above

$$E\left[Y(t) \cdot Y(s)\right] = E\left[2tW\left(\frac{1}{4t}\right) \cdot 2sW\left(\frac{1}{4s}\right)\right] = 4ts\min\left(\frac{1}{4t}, \frac{1}{4s}\right) = 4ts\frac{1}{4t} = s,$$

and if $s > t$

$$E\left[Y(t) \cdot Y(s)\right] = 4ts\frac{1}{4s} = t.$$
$$E\left[Y(t) \cdot Y(s)\right] = \min(t, s).$$

Thus $\{Y(t) \mid 0 \le t\}$ is a Gaussian process with mean function $= 0$ and autocovariance function $\text{Cov}_Y(t, s) = \min(t, s)$. Hence $\{Y(t) \mid 0 \le t\}$ is a Wiener process, and has as such independent increments.

Uppgift 5

a) Are the sample paths of $\mathbf{Y}$ nondecreasing ? No, they are not. Assume $t < T_1 < 1$, i.e. the first arrival is after t and before $t = 1$. Then $N(t) = 0$ for $t < T_1$, $N(1) \ge 1$ and the sample path of the process is nothing but $-tN(1)$, which is a decreasing function of t.

ANSWER a): No ! .

b)

$$E\left[Y(t)\right] = E\left[N(t) - tN(1)\right] = E\left[N(t)\right] - tE\left[N(1)\right]$$
$$= \lambda t - \lambda t = 0,$$

since $N(t) \in \text{Po}(\lambda t)$ and $N(1) \in \text{Po}(\lambda)$.

ANSWER b): $\underline{E\left[Y(t)\right] = 0}$.

c)

$$\text{Var}\left[Y(t)\right] = \text{Var}\left[N(t)\right] + t^2\text{Var}\left[N(1)\right] - 2t\text{Cov}(N(t), N(1))$$
$$= \lambda t + \lambda t^2 - 2t\lambda t,$$

since $\text{Cov}_{\mathbf{N}}(t, 1) = \lambda\min(t, 1) = \lambda t$,

$$= \lambda t - \lambda t^2 = \lambda t(1 - t).$$

ANSWER c): $\underline{\text{Var}\left[Y(t)\right] = \lambda t(1 - t)}, \quad 0 \le t \le 1.$

d) The autocovariance function of $\mathbf{Y}$ follows by

$$\text{Cov}_{\mathbf{Y}}(t,s) = \text{Cov}(N(t) - tN(1), N(s) - sN(1))$$

$$= \text{Cov}(N(t), N(s)) - s\text{Cov}(N(t), N(1)) - t\text{Cov}(N(1), N(s)) + ts\text{Cov}(N(1), N(1)),$$

where we used a formula in section 2.5. in the Collection of Formulas. Furthermore, from the expression $\text{Cov}_{\mathbf{N}}(t,s) = \lambda \min(t,s)$ we get

$$= \lambda \min(t,s) - s\lambda t - t\lambda s + ts\lambda = \lambda \min(t,s) - 2\lambda ts + \lambda ts = \lambda \min(t,s) - \lambda ts$$

$$= \begin{cases} \lambda s(1-t) & s < t, \\ \lambda t(1-s) & t \leq s. \end{cases}$$

$$\text{ANSWER b): } \underline{\text{Cov}_{\mathbf{N}}(t,s) = \begin{cases} \lambda s(1-t) & s < t, \\ \lambda t(1-s) & t \leq s. \end{cases}}.$$

Uppgift 6

a)

$$\varphi_{S_N}(t) = E\left[e^{itS_N}\right] = E\left[E\left[e^{itS_N} \mid N\right]\right]$$

and with $H(N) = E\left[e^{itS_N} \mid N\right]$ (Doob-Dynkin Theorem)

$$= E\left[H(N)\right] = \sum_{n=0}^{\infty} H(n)\mathbf{P}(N = n)$$

$$= \sum_{n=0}^{\infty} E\left[e^{itS_N} \mid N = n\right]\mathbf{P}(N = n) = \sum_{n=0}^{\infty} E\left[e^{itS_n} \mid N = n\right]\mathbf{P}(N = n) = \sum_{n=0}^{\infty} E\left[e^{itS_n}\right]\mathbf{P}(N = n),$$

where we used the independence of N and the X_is,

$$= \sum_{n=0}^{\infty} E\left[e^{itX_1+X_2+\ldots+X_n}\right]\mathbf{P}(N = n)$$

$$= \sum_{n=0}^{\infty} \left(\varphi_X(t)\right)^n \mathbf{P}(N = n) = g_N\left(\left(\varphi_X(t)\right)\right),$$

which is the composition formula that was to be shown.

b) If $\{X_n\}_{n\geq 1}$ is a sequence of I.I.D. r.v.'s such that

$$\underline{\text{P}}\left(X_n = x\right) = \begin{cases} \frac{1}{2} & x = -1 \\ \frac{1}{2} & x = 1, \end{cases}$$

then

$$\varphi_X(t) = E\left[e^{itX}\right] = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos(t),$$

where we used Euler's formula for $\cos(t)$. By the Collection of Formulas we get for $N \in \text{Po}(\lambda)$ the probability generating function $g_N(t) = e^{\lambda(t-1)}$ and by a) we get

$$\varphi_{S_N}(t) = g_N\left(\left(\varphi_X(t)\right) = e^{\lambda(\varphi_X(t)-1)}\right.$$

$$= e^{\lambda(\cos(t)-1} = e^{-\lambda(1-\cos(t))}.$$

or

$$\varphi_{S_N}(t) = e^{-\lambda(1-\cos(t))}.$$

c) We find the characteristic function of $\frac{S_N}{\sqrt{\lambda}}$. By c) we have

$$\varphi_{\frac{S_N}{\sqrt{\lambda}}}(t) = \varphi_{S_N}\left(\frac{t}{\sqrt{\lambda}}\right) =$$

$$= e^{-\lambda\left(1-\cos\left(\frac{t}{\sqrt{\lambda}}\right)\right)}.$$

on p. 198 (section 8.6) of L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering* yields

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - -\frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{2n!} + \ldots$$

so that

$$\cos\left(\frac{t}{\sqrt{\lambda}}\right) = 1 - \frac{t^2}{2!\lambda} + \frac{t^4}{4!\lambda^2} - \frac{t^6}{6!\lambda^3} + \ldots + (-1)^n \frac{t^{2n}}{2n!\lambda^n} + \ldots$$

Thus

$$\lambda\left(1 - \cos\left(\frac{t}{\sqrt{\lambda}}\right)\right) = \frac{t^2}{2!} - \frac{t^4}{4!\lambda} + \ldots,$$

and we get

$$\varphi_{\frac{S_N}{\sqrt{\lambda}}}(t) = e^{-\frac{t^2}{2!} + \frac{t^4}{4!\lambda} + \cdots} \to e^{-\frac{t^2}{2}}$$

as $\lambda \to \infty$. Since $e^{-\frac{t^2}{2}}$ is the characteristic function of $N(0,1)$ and is continuous at $t = 0$, we have shown that

$$\frac{S_N}{\sqrt{\lambda}} \xrightarrow{d} N(0,1).$$

as $\lambda \to \infty$.

d) We have that

$$E[S_N] = E[N]\, E[X] = 0,$$

as $E[X] = 0$. In addition $\mathrm{Var}[X] = 1$, and $E[N] = \lambda$. Hence λ is the expected number of terms in S_N. Hence the pattern of CLT, $\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}(X_i - \mu)$ is roughly the same as in $\frac{S_N}{\sqrt{\lambda}}$.