Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY THE 9^{th} OF JANUARY 2013 02.00 p.m.–07.00 p.m.

Examinator: Timo Koski, tel. 790 71 34, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with $27-29$ points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 9^{th} of January 2013 at 07.15 p.m..

The exam results will be announced at the latest on Friday the 25^{th} of January 2013.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

Let $(\Omega, \mathcal{F})$ be a measurable space and let $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ designate the real line and its Borel sigma field, respectively.

a) Give the definition of a real valued random variable X in $(\Omega, \mathcal{F})$. (1 p)

b) Give the definition of a Borel function f. (2 p)

c) Let X be a random variable, and f be a Borel function. Set $Y = f(X)$. Prove that Y is a random variable in $(\Omega, \mathcal{F})$. *Aid:* Use the definitions a) and b). (7 p)

Uppgift 2

$X \in \text{Exp}(1)$, $Y \in \text{Exp}(1)$. X and Y are indepedent. Find the probability

$$\mathbf{P}(X \leq Y).$$

Please show your computations. (10 p)

Uppgift 3

X and Y are two independent continuous random variables. Their probability density functions are $f_X(x)$ and $f_Y(y)$, respectively.

a) Find the probability density function $f_U(u)$ of the ratio $U = \frac{X}{Y}$. *Aid:* The trick is to consider the change of variables $U = \frac{X}{Y}$ and $V = Y$, i.e., V is an auxiliary variable. (5 p)

b) Show that if $X \in N(0,1)$ and $Y \in N(0,1)$ and X and Y are independent, then $\frac{X}{Y} \in C(0,1)$. (5 p)

Uppgift 4

Consider the Wiener process $\{W(t) \mid t \geq 0\}$. Introduce

$$Y(t) = \frac{W(t) - W(t/2)}{\sqrt{t}} \quad t > 0.$$

a) Find the distribution of $Y(t)$. (4 p)

b) Find the autocorrelation function of the process $\{Y(t) \mid t > 0\}$. (4 p)

c) Is the process $\{Y(t) \mid t > 0\}$ weakly stationary ? Is the process $\{Y(t) \mid t > 0\}$ a Wiener process ? Justify your answers. (2 p)

(10 p)

Uppgift 5

$\mathbf{N} = \{N(t) \mid t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$. Show that for $t > s > 0$ there is the conditional distribution

$$N(s) \mid N(t) = y \in \text{Bin}\left(y, \frac{s}{t}\right).$$

Aid: Compute first $\mathbf{P}\left(N(s) = x, N(t) = y\right)$ with $y \geq x \geq 0$.

(10 p)

Uppgift 6

$\{X_n\}_{n\geq 1}$ is a sequence of indepedent and identically distributed r.v.'s. Their common mean is $\mu \neq 0$ and their common variance is $0 < \sigma^2 < +\infty$. Set $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Show that

$$\sqrt{n}\frac{S_n - n\mu}{S_n + n\mu} \xrightarrow{d} N\left(0, a^2\right),$$

as $n \to +\infty$, and determine a^2. Please justify your solution in detail. (10 p)

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SOLUTIONS TO THE EXAM SATURDAY THE 9^{th} OF JANUARY 2013.

Uppgift 1

a) A real valued random variable is a real valued function $X : \Omega \to \mathbf{R}$ such that for every set $A \in \mathcal{B}$, the Borel σ algebra over $\mathbf{R}$,

$$X^{-1}(A) = \{\omega : X(\omega) \in A\} \in \mathcal{F}.$$

b) A function $f : \mathbf{R} \mapsto \mathbf{R}$ is called a Borel function, if for every set A in $\mathcal{B}$, the Borel σ algebra, we have that

$$f^{-1}(A) = \{x \in \mathbf{R} \mid f(x) \in A\}$$

belongs to the Borel σ algebra, i.e.,

$$f^{-1}(A) \in \mathcal{B}.$$

c) Let A be a Borel set, i.e., $A \in \mathcal{B}$. We consider

$$Y^{-1}(A) = \{\omega \in \Omega \mid Y(\omega) \in A\}.$$

By construction we have $Y(\omega) = f(X(\omega))$, and thus

$$Y^{-1}(A) = \{\omega \in \Omega \mid f(X(\omega)) \in A\} = \{\omega \in \Omega \mid X(\omega) \in f^{-1}(A)\},$$

where the inverse image is $f^{-1}(A) = \{x \in \mathbf{R} \mid f(x) \in A\}$. Since f is a Borel function, we have by definition that $f^{-1}(A) \in \mathcal{B}$, since $A \in \mathcal{B}$. But then

$$\{\omega \in \Omega \mid X(\omega) \in f^{-1}(A)\} \in \mathcal{F},$$

since X is a random variable. But thereby we have established that $Y^{-1}(A) \in \mathcal{F}$ for any A in $\mathcal{B}$, which by definition means that Y is a random variable.

Uppgift 2

We write

$$\mathbf{P}(X \leq Y) = \mathbf{P}(X - Y \leq 0),$$

and determine the distribution of $X - Y$ using the characteristic function. As X and Y are assumed independent we get

$$\varphi_{X-Y}(t) = \varphi_X(t) \cdot \varphi_{-Y}(t) = \varphi_X(t) \cdot \varphi_Y(-t),$$

and by the Appendix B of Gut

$$= \frac{1}{1-it} \cdot \frac{1}{1-i(-t)} = \frac{1}{1-it} \cdot \frac{1}{1+it}$$
$$= \frac{1}{1+t^2}.$$

Here a reference to Appendix B of Gut gives that $X - Y \in L(1)$. However, since $\varphi_{X-Y}(t)$ is a real function, we get $X - Y \stackrel{d}{=} Y - X$. In other words

$$\mathbf{P}(X - Y \leq 0) = \mathbf{P}(Y - X \leq 0) = \frac{1}{2}.$$

ANSWER a): $\underline{\mathbf{P}(X \leq Y) = \frac{1}{2}.}$

Uppgift 3

a) We find the joint p.d.f., here $f_{U,V}(u, v)$, and marginalize to U from that the desired p.d.f..

The inverse map is found as

$$X = h_1(U, V) = UV, \quad Y = h_2(U, V) = V.$$

Then the Jacobian is

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

Then

$$f_{U,V}(u, v) = f_X(uv) f_Y(v) |v|.$$

Hence the distribution of the ratio $U = \frac{X}{Y}$ is given by the marginal density

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv.$$

This can be written as

$$f_U(u) = \int_0^{\infty} f_X(uv) f_Y(v)\, v dv - \int_{-\infty}^{0} f_X(uv) f_Y(v)\, v dv. \tag{1}$$

ANSWER a): $\underline{f_U(u) = \int_0^{\infty} f_X(uv) f_Y(v)\, v dv - \int_{-\infty}^{0} f_X(uv) f_Y(v)\, v dv.}$

b) If $X \in N(0, 1)$ and $Y \in N(0, 1)$ and X and Y are independent, then (1) gives for any $u \in (-\infty, \infty)$

$$f_U(u) = \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{1}{2}u^2v^2} e^{-\frac{1}{2}v^2} v dv - \frac{1}{2\pi} \int_{-\infty}^{0} e^{-\frac{1}{2}u^2v^2} e^{-\frac{1}{2}v^2} v dv.$$

We compute the first integral in the right hand side.

$$\frac{1}{2\pi} \int_0^{\infty} e^{-\frac{1}{2}u^2v^2} e^{-\frac{1}{2}v^2} v dv = \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{1}{2}v^2\left(1+u^2\right)} v dv$$

$$= \frac{1}{2\pi}\left[\frac{-1}{1+u^2}e^{-\frac{1}{2}v^2\left(1+u^2\right)}\right]_{v=0}^{\infty} = \frac{1}{2\pi}\frac{1}{1+u^2}.$$

In the same manner we get

$$\frac{1}{2\pi}\int_{-\infty}^{0} e^{-\frac{1}{2}u^2v^2}e^{-\frac{1}{2}v^2}vdv = \frac{1}{2\pi}\frac{-1}{1+u^2}.$$

Thus

$$\frac{1}{2\pi}\int_{0}^{\infty} e^{-\frac{1}{2}u^2v^2}e^{-\frac{1}{2}v^2}vdv - \frac{1}{2\pi}\int_{-\infty}^{0} e^{-\frac{1}{2}u^2v^2}e^{-\frac{1}{2}v^2}vdv = \frac{1}{2\pi}\frac{1}{1+u^2} - \frac{1}{2\pi}\frac{-1}{1+u^2} = \frac{1}{\pi}\frac{1}{1+u^2}.$$

We have found

$$f_U(u) = \frac{1}{\pi}\frac{1}{1+u^2}, \quad -\infty < u < \infty.$$

This is the probability density function of C(0, 1), as was to be shown.

Uppgift 4

a) Since $\{W(t) \mid t \geq 0\}$ is a Wiener process

$$Y(t) = \frac{W(t) - W(t/2)}{\sqrt{t}} \quad t > 0$$

is a linear combination of two Gaussian random variables and thus a Gaussian random variable. Since $E[W(t)] = 0$, we get $E[Y(t)] = 0$. By properties of the Wiener process (Colletion of Formulas), $Z = W(t) - W(t/2) \in N(0, t/2)$. Then $Y(t) = \frac{Z}{\sqrt{t}}$ and

$$\mathrm{Var}[Y(t)] = \frac{1}{t}\mathrm{Var}[Z] = \frac{1}{t}\cdot\frac{t}{2} = \frac{1}{2}.$$

ANSWER a): $\underline{Y(t) \in N\left(0, \frac{1}{2}\right), t > 0.}$

b) The autocorrelation function of the process $\{Y(t) \mid t > 0\}$ is given by

$$R_Y(t,s) = E[Y(t)Y(s)] = \frac{1}{\sqrt{ts}}E[(W(t) - W(t/2))(W(s) - W(s/2))]$$

$$= \frac{1}{\sqrt{ts}}\left(E[(W(t)W(s)] - E[(W(t)W(s/2)] - E[(W(t/2)W(s)] + E[(W(t/2)W(s/2)]\right).$$

$$= \frac{1}{\sqrt{ts}}\left(\min(t,s) - \min(t,s/2) - \min(t/2,s) + \min(t/2,s/2)\right).$$

Here we must distinguish between four different cases.

1. $t > s$

   1.1 $t > s > t/2$. Then

$$R_Y(t,s) = \frac{1}{\sqrt{ts}}\left(s - s/2 - t/2 + s/2\right) = \frac{1}{\sqrt{ts}}\left(s - t/2\right).$$

1.2 $t > t/2 > s$. Then

$$R_Y(t,s) = \frac{1}{\sqrt{ts}}\left(s - s/2 - s + s/2\right) = 0.$$

2. $s > t$

2.1 $s > t > s/2$. Then

$$R_Y(t,s) = \frac{1}{\sqrt{ts}}\left(t - s/2\right).$$

2.2 $s > s/2 > t$. Then

$$R_Y(t,s) = \frac{1}{\sqrt{ts}}\left(t - t - t/2 + t/2\right) = 0.$$

ANSWER b): $\underline{1.1 - 1.2 \quad \& \quad 2.1 - 2.2 \quad \text{above.}}$

c) The process $\{Y(t) \mid t > 0\}$ is not weakly stationary, as its autocorrelation function $R_Y(t,s)$ is not a function of $|t-s|$. The process $\{Y(t) \mid t > 0\}$ is not a Wiener process, e.g., it has not almost surely the value $= 0$ at $t = 0$. Also, $Y(t) \in N\left(0, \frac{1}{2}\right) \neq N(0,t)$.

Uppgift 5

Since $\mathbf{N} = \{N(t) \mid t \geq 0\}$ is a Poisson process we get for $t > s$ and $y \geq x$

$$\mathbf{P}\left(N(s) = x, N(t) = y\right) = \mathbf{P}\left(N(s) = x, N(t) - N(s) = y - x\right)$$

$$= \mathbf{P}\left(N(s) = x\right)\mathbf{P}\left(N(t) - N(s) = y - x\right),$$

since the increments are independent. As $\mathbf{N}$ is a Poisson process with intensity $\lambda > 0$ we have

$$\mathbf{P}\left(N(s) = x\right) = e^{-\lambda s}\frac{(\lambda s)^x}{x!},$$

and

$$\mathbf{P}\left(N(t) - N(s) = y - x\right) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{y-x}}{(y-x)!}.$$

Then

$$\mathbf{P}\left(N(s) = x, N(t) = y\right) = \frac{(\lambda s)^x(\lambda(t-s))^{y-x}e^{-\lambda t}}{x!(y-x)!}.$$

Then

$$\mathbf{P}\left(N(s) = x | N(t) = y\right) = \frac{\mathbf{P}\left(N(s) = x, N(t) = y\right)}{\mathbf{P}\left(N(t) = y\right)} = \frac{\frac{(\lambda s)^x(\lambda(t-s))^{y-x}e^{-\lambda t}}{x!(y-x)!}}{e^{-\lambda t}\frac{(\lambda t)^y}{y!}}$$

$$= \frac{y!}{x!(y-x)!}\frac{(\lambda s)^x}{(\lambda t)^x}\frac{(\lambda t)^x}{(\lambda t)^y}(\lambda(t-s))^{y-x}$$

$$= \frac{y!}{x!(y-x)!}\left(\frac{s}{t}\right)^x\frac{(\lambda t)^x}{(\lambda t)^y}(\lambda(t-s))^{y-x}$$

$$= \frac{y!}{x!(y-x)!}\left(\frac{s}{t}\right)^x\frac{(\lambda(t-s))^{y-x}}{(\lambda t)^{y-x}}$$

$$= \frac{y!}{x!(y-x)!} \left(\frac{s}{t}\right)^x \left(1 - \frac{s}{t}\right)^{y-x}.$$

$$= \begin{pmatrix} y \\ x \end{pmatrix} \left(\frac{s}{t}\right)^x \left(1 - \frac{s}{t}\right)^{y-x}.$$

This is the probability mass function $p_X(x)$ of $X \in \text{Bin}\left(y, \frac{s}{t}\right)$.

Uppgift 6

We write $S_n - n\mu = \sum_{i=1}^{n}(X_i - \mu)$. Then

$$\sqrt{n}\frac{S_n - n\mu}{S_n + n\mu} = \sqrt{n}\frac{\sum_{i=1}^{n}(X_i - \mu)}{S_n + n\mu}$$

$$= \frac{n}{\sqrt{n}} \frac{\sum_{i=1}^{n}(X_i - \mu)}{S_n + n\mu} = \frac{1}{\sqrt{n}} \frac{\sum_{i=1}^{n}(X_i - \mu)}{\frac{1}{n}(S_n + n\mu)}.$$

We take first a look at the numerator.

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) = \sigma \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu).$$

By the **central limit theorem**

$$\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \xrightarrow{d} N(0,1),$$

as $n \to +\infty$.
In the denominator

$$\frac{1}{n}(S_n + n\mu) = \frac{1}{n}S_n + \mu \xrightarrow{P} \mu + \mu = 2\mu,$$

$n \to +\infty$, since the **weak law of large numbers** entails $\frac{1}{n}S_n \xrightarrow{P} \mu$, as $n \to +\infty$, where we also applied one of the cases of **Cramér - Slutzky**.
Then the **Cramér - Slutzky** theorem yields

$$\sqrt{n}\frac{S_n - n\mu}{S_n + n\mu} = \frac{1}{\sqrt{n}} \frac{\sum_{i=1}^{n}(X_i - \mu)}{\frac{1}{n}(S_n + n\mu)} \xrightarrow{P} \frac{\sigma}{2\mu} Z,$$

as $n \to +\infty$, where $Z \in N(0,1)$. Hence, if $a^2 = \frac{\sigma^2}{4\mu^2}$, we have

$$\sqrt{n}\frac{S_n - n\mu}{S_n + n\mu} \xrightarrow{P} N(0, a^2),$$

as $n \to +\infty$.

ANSWER: $\underline{a^2 = \frac{\sigma^2}{4\mu^2}}$.