Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY TUESDAY THE 29^{th} OF OCTOBER 2013 08.00 a.m.–01.00 p.m.

Examinator: Timo Koski, tel. 070 2370047, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Tuesday 29^{th} of October 2013 at 1.15 p.m..

The exam results will be announced at the latest on Tuesday the 12^{th} of November, 2013.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

a) Let A, B, C be events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $A \cap B \subseteq C$. Show that

$$\mathbf{P}(C^c) \leq \mathbf{P}(A^c) + \mathbf{P}(B^c)$$

(3 p)

b) Let $\{X_n\}_{n\geq 1}$ be a sequence of r.v.'s in a probability space and X and Y be two random variables in the same space. Take $\epsilon > 0$ and let $C = \{|X - Y| \leq \epsilon\}$, $A = \{|X_n - X| \leq \epsilon/2\}$ and $B = \{|X_n - Y| \leq \epsilon/2\}$.

Check that $A \cap B \subseteq C$. (3 p)

c) Assume next that $X_n \overset{P}{\to} X$, as $n \to \infty$ and that $X_n \overset{P}{\to} Y$, as $n \to \infty$. What do you now know about X and Y in view of a) and b)? (4 p)

Uppgift 2

The random variables X_1, X_2,... are independent and $\in N(0, 1)$. Set

$$N \overset{\text{def}}{=} \min\{k | X_k \geq 0\}$$

a) Show that the probability density function of X_N is

$$f_{X_N}(x) = \begin{cases} 2\phi(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Aid: It may be helpful to compute the probability $\mathbf{P}(X_N \leq x \mid N = n)$. (5 p)

b) Find $E[X_N]$ and $\text{Var}[X_N]$. . (5 p)

Uppgift 3

We say that a random variable X is Linnik(α)-distributed, $2 \geq \alpha > 0$, if its characteristic function is given by

$$\varphi_X(t) = \frac{1}{1 + |t|^\alpha}.$$

Let X_1, X_2,... be independent and identically Linnik(α)-distributed,
Let $N \in \text{FS}(p)$, and let N be independent of X_1, X_2,.... Set

$$S_N \overset{\text{def}}{=} X_1 + X_2 + \ldots + X_N.$$

Find the distribution of

$$p^{1/\alpha} S_N.$$

(10 p)

Uppgift 4

Use a central limit theorem for suitably chosen Poisson random variables to prove that

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}=\frac{1}{2}.$$

(10 p)

Uppgift 5

$X=\{X(t)\mid -\infty<t<\infty\}$ is a Gaussian stochastic process. Its mean function is $\mu(t)=2$ for all t and its autocovariance function is

$$\mathrm{Cov}\left[X(t),X(s)\right]=C_X(h)=\max\left(0,1-\frac{|h|}{2}\right),\quad h=t-s.$$

a) Find the distribution of the bivariate random variable

$$(X(t)-X(t-0.2),X(t-0.1))^T.$$

(3 p)

b) Why does it hold that

$$(X(t)-X(t-0.2),X(t-0.1))^T\stackrel{d}{=}(X(t+1)-X(t+0.8),X(t+0.9))^T?$$

(1 p)

c) Find the probability $\mathbf{P}\left(X(t)>X(t-0.2)+1|X(t-0.1)=1\right)$. (2 p)

d) We sample the process X in time so that our samples are

$$X_k=X(2k),\quad k=1,2,\ldots.$$

Show that

$$\frac{1}{n}\sum_{k=1}^{n}X_k\xrightarrow{P}a,$$

as $n\to+\infty$ and determine the limiting value a. Be so kind and justify Your steps of solution very carefully. (4 p)

Uppgift 6

$\mathbf{W}=\{W(t)\mid t\geq 0\}$ is a Wiener process, $h(u)$ is a function such that $\int_0^\infty h^2(u)du<\infty$. Let for any $t>0$

$$Y(t)=e^{\int_0^t h(u)dW(u)}.$$

Find the autocorrelation function of the process $\mathbf{Y}=\{Y(t)\mid t\geq 0\}$ as

$$R_{\mathbf{Y}}(t,s)=\begin{cases} e^{\frac{1}{2}\int_s^t h^2(u)du+2\int_0^s h^2(u)du} & t>s\\ e^{\frac{1}{2}\int_t^s h^2(u)du+2\int_0^t h^2(u)du} & t\leq s.\end{cases}$$

Aid: The rule of double expectation may turn out to be useful to start with. (10 p)

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SOLUTIONS TO THE EXAM TUESDAY THE 29^{th} OF OCTOBER 2013.

Uppgift 1

a) If $A \cap B \subseteq C$, then it holds for the complement sets that $C^c \subseteq (A \cap B)^c$. By De Morgan's law (Beta) we get $(A \cap B)^c = A^c \cup B^c$. Thus by properties of any probability measure (the first inequality in Collection of formulas)

$$\mathbf{P}\left(C^c\right) \leq \mathbf{P}\left(A^c \cup B^c\right)$$

and by Boole's inequality (Beta and the Collection of Formulas)

$$\mathbf{P}\left(A^c \cup B^c\right) \leq \mathbf{P}\left(A^c\right) + \mathbf{P}\left(B^c\right),$$

which establishes the assertion as claimed.

b) For given $\epsilon > 0$ we have $C = \{|X - Y| \leq \epsilon\}$, $A = \{|X_n - X| \leq \epsilon/2\}$ and $B = \{|X_n - Y| \leq \epsilon/2\}$. We need to check that

$$A \cap B \subseteq C,$$

We get by the triangle inequality that

$$|X - Y| = |(X - X_n) + (X_n - Y)| \leq |X - X_n| + |X_n - Y|$$

But $A \cap B$ is the event that both $A = \{|X_n - X| \leq \epsilon/2\}$ and $B = \{|X_n - Y| \leq \epsilon/2\}$ hold. Hence if the event $A \cap B$ holds,

$$|X - X_n| + |X_n - Y)| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

i.e., if the event $A \cap B$ holds, then

$$|X - Y| \leq \epsilon.$$

Thus it holds that $A \cap B \subseteq C$.

c) $X_n \stackrel{P}{\to} X$, as $n \to \infty$ and that $X_n \stackrel{P}{\to} Y$, as $n \to \infty$ mean that

$$\mathbf{P}\left(A^c\right) = \mathbf{P}\left(\{|X_n - X| > \epsilon/2\}\right) \to 0$$

and

$$\mathbf{P}\left(B^c\right) = \mathbf{P}\left(\{|X_n - Y| > \epsilon/2\}\right) \to 0$$

as $n \to \infty$. Hence the inequality in a), which can be used in view of b), means that

$$\mathbf{P}\left(C^c\right) = \mathbf{P}\left(\{|X - Y| > \epsilon\}\right) = 0$$

for any $\epsilon > 0$. Hence we have shown that

$$\mathbf{P}\left(X \neq Y\right) = 0$$

i.e., that **the limiting random variable for convergence in probability is almost surely unique**.

Uppgift 2

a) We express

$$\mathbf{P}\left(X_N \leq x \mid N = n\right)$$

as

$$= \mathbf{P}\left(X_n \leq x \mid X_1 < 0, \ldots, X_{n-1} < 0, X_n \geq 0\right),$$

and by independence we get

$$= \mathbf{P}\left(X_n \leq x \mid X_n \geq 0\right),$$

and as the r.v.'s are I.I.D.,

$$= \mathbf{P}\left(X_1 \leq y \mid X_1 \geq 0\right)$$

and by definition of conditional probability

$$= \frac{\mathbf{P}\left(0 \leq X_1 \leq y\right)}{\mathbf{P}\left(X_1 \geq 0\right)}$$

$$= \frac{\Phi(x) - \Phi(0)}{\frac{1}{2}}$$

$$= 2 \cdot \left(\Phi(x) - \Phi(0)\right) \quad x \geq 0.$$

Thus X_N is independent of N and we get

$$f_{X_N}(x) = \frac{d}{dx}\mathbf{P}\left(X_N \leq x \mid N = n\right) = \frac{d}{dx}\mathbf{P}\left(X_N \leq x\right) = 2\phi(x) I\left(x \geq 0\right).$$

ANSWER a): $\underline{f_{X_N}(x) = 2\phi(x), x \geq 0, f_{X_N}(x) = 0, x < 0.}$

b)

$$E\left[X_N\right] = \int_0^{+\infty} x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}}\left[-e^{-x^2/2}\right]_0^{\infty} = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}.$$

Then by the symmetry property of $x^2\phi(x)$

$$E\left[X_N^2\right] = \int_0^{+\infty} x^2 2\phi(x) dx = 2\int_0^{+\infty} x^2\phi(x) dx = \int_{-\infty}^{+\infty} x^2\phi(x) dx = 1.$$

Hence

$$\mathrm{Var}(X_N) = E\left[X_N^2\right] - \left(E\left[X_N\right]\right)^2 = 1 - \frac{2}{\pi}.$$

as was to be found.

Uppgift 3

We use the composition formula to represent the characteristic function of the sum S_N as

$$\varphi_{S_N}(t) = g_N\left(\varphi_X(t)\right),$$

where $g_N(t)$ is the probability generating function of N. Since $N \in \mathrm{FS}(p)$ we have by definition and by Appendix B,

$$g_N(t) = \sum_{k=0}^{\infty} t^k p_N(k) = \sum_{k=1}^{\infty} t^k p(1-p)^{k-1} =$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} t^k (1-p)^k = \frac{p}{1-p} \left(\sum_{k=0}^{\infty} t^k (1-p)^k - 1 \right)$$

$$= \frac{p}{1-p} \left(\frac{1}{1-(1-p)t} - 1 \right) = \frac{p}{1-p} \frac{1-(1-(1-p)t)}{1-(1-p)t} =$$

$$= \frac{pt}{1-(1-p)t},$$

where we require that $|(1-p)t| < 1$.

> This p.g.f. is given directly in Råde- Westergren: Beta, chapter 17.2, if one observes that FS(p) is called geometric G(p) in loc.cit,

Then the composition formula gives

$$\varphi_{S_N}(t) = g_N\left(\varphi_X(t)\right) = \frac{p\varphi_X(t)}{1-(1-p)\varphi_X(t)}$$

$$= \frac{p\frac{1}{1+|t|^\alpha}}{1-(1-p)\frac{1}{1+|t|^\alpha}}.$$

Then we get the characteristic function of $p^{1/\alpha} S_N$ as

$$\varphi_{p^{1/\alpha}S_N}(t) = \varphi_{S_N}(p^{1/\alpha}t) = \frac{p\frac{1}{1+p|t|^\alpha}}{1-(1-p)\frac{1}{1+p|t|^\alpha}}$$

$$= \frac{p}{\left(\frac{1+p|t|^\alpha-(1-p)}{1+p|t|^\alpha}\right)} \left(\frac{1}{1+p|t|^\alpha}\right) = \frac{p}{p+p|t|^\alpha} = \frac{1}{1+|t|^\alpha}.$$

By uniqueness of the characteristic function we get that $p^{1/\alpha} S_N$ is Linnik(α)-distributed.

$$\text{ANSWER} : \underline{p^{1/\alpha} S_N \in \mathrm{Linnik}\,(\alpha)\,.}$$

Uppgift 4

Let $U_i \in \mathrm{Po}(1)$ be independent, for $i = 1, 2, \ldots,$. Set

$$X_n = U_1 + \ldots U_n.$$

Then it follows by identical Po(1) -distributions that

$$E\left[U_i\right] = 1, \mathrm{Var}\left[U_i\right] = 1,$$

and in addition we know (see Beta) that

$$X_n \in \mathrm{Po}(n).$$

and thus $P(X_n = k) = e^{-n}\frac{n^k}{k!}$. We get thereby that

$$e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = P(X_n \leq n) = P(X_n - n \leq 0) =$$

$$= P\left(\frac{X_n - n}{\sqrt{n}} \leq 0\right).$$

We write

$$\frac{X_n - n}{\sqrt{n}} = \frac{\sum_{i=0}^{n}(U_i - 1)}{\sqrt{n}}$$

Then we get by the **central limit theorem**, that

$$\frac{\sum_{i=0}^{n}(U_i - 1)}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0,1),$$

as $n \to \infty$. By definition, the convergence in distribution means that

$$P\left(\frac{X_n - n}{\sqrt{n}} \leq 0\right) = P\left(\frac{\sum_{i=0}^{n}(U_i - 1)}{\sqrt{n}} \leq 0\right) = F_{\frac{\sum_{i=0}^{n}(U_i-1)}{\sqrt{n}}}(0) \to \Phi(0) = \frac{1}{2}.$$

This is as asserted.

(10 p)

Uppgift 5

a) We set

$$Y_1 = X(t) - X(t-0.2), Y_2 = X(t-0.1)$$

and write this as with a vector notation

$$\mathbf{Y} \stackrel{\mathrm{def}}{=} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = B\mathbf{X}$$

where

$$\mathbf{X} \stackrel{\mathrm{def}}{=} \begin{pmatrix} X(t) \\ X(t-0.1) \\ X(t-0.2) \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $\mathbf{X}$ is a vector with a multivariate normal distribution (as the process X is Gaussian), the distribution of $\mathbf{Y}$ is the multivariate normal distribution (Collection of Formulas)

$$N\left(B\mu, BCB^T\right)$$

where μ is the mean vector of $\mathbf{X}$ and C is the covariance matrix of $\mathbf{X}$. From the information about the process X we get that

$$\mu = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 19/20 & 18/20 \\ 19/20 & 1 & 19/20 \\ 18/20 & 19/20 & 1 \end{pmatrix}.$$

Then matrix computations give

$$B\mu = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad BCB^T = \begin{pmatrix} 0.2 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{ANSWER} : \underline{\begin{pmatrix} X(t) - X(t-0.2) \\ X(t-0.1) \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.2 & 0 \\ 0 & 1 \end{pmatrix}\right).}$$

b) Since the process X is Gaussian and weakly stationary (=constant mean function, and the a.c.f. depends only on the difference between time instants), it is also (strictly) stationary. Therefore

$$\begin{pmatrix} X(t) \\ X(t-0.1) \\ X(t-0.2) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X(t+1) \\ X(t+1-0.1) \\ X(t+1-0.2) \end{pmatrix} = \begin{pmatrix} X(t+1) \\ X(t+0.9) \\ X(t+0.8) \end{pmatrix}$$

Since

$$\begin{pmatrix} X(t+1) - X(t+0.8) \\ X(t+0.9) \end{pmatrix} = B \begin{pmatrix} X(t+1) \\ X(t+0.9) \\ X(t+0.8) \end{pmatrix}$$

the desired conclusion follows.

c) When we use the notations from part a), the probability $\mathbf{P}\left(X(t) > X(t-0.2) + 1 | X(t-0.1) = 1\right)$ can be written as

$$\mathbf{P}\left(Y_1 > 1 | Y_2 = 1\right)$$

However, the covariance matrix in a) shows that Y_1 and Y_2 are independent as they are non-correlated Gaussians. We see also that $Y_1 \in N(0, 0.2)$. Hence we get

$$\mathbf{P}\left(Y_1 > 1 | Y_2 = 1\right) = \mathbf{P}\left(Y_1 > 1\right) = \mathbf{P}\left(\frac{Y_1}{\sqrt{0.2}} > \frac{1}{\sqrt{0.2}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{0.2}}\right),$$

as $\frac{Y_1}{\sqrt{0.2}} \in N(0,1)$.

$$\text{ANSWER} : \underline{\mathbf{P}\left(X(t) > X(t-0.2) + 1 | X(t-0.1) = 1\right) = 1 - \Phi\left(\tfrac{1}{\sqrt{0.2}}\right).}$$

d) If the samples of the process X in time are

$$X_k = X(2k), \quad k = 1, 2, \ldots, .$$

then for every k

$$E[X_k] = 2$$

and

$$\mathrm{Cov}(X_k, X_l) = \mathrm{Cov}[X(2k), X(2l)] = C_X(2(k-l)) = \max(0, 1 - |k-l|).$$

Thus

$$\mathrm{Cov}(X_k, X_l) = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

Thus X_ks are Gaussian and non correlated, therefore independent r.v.'s and the standard weak law of large numbers shows that

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{P} 2,$$

as $n \to +\infty$, since $E[X_k] = 2$ for all k.

Uppgift 6

By definition we have

$$R_{\mathbf{Y}}(t,s) = E[Y(t) \cdot Y(s)] = E\left[e^{\int_0^t h(u)dW(u)} \cdot e^{\int_0^s h(u)dW(u)}\right]$$

We assume $t > s > 0$ and set $\mathcal{F}_s^{\mathbf{W}}$ equal to the sigma field generated by $\mathbf{W}$ up to time s. Then double expectation gives

$$= E\left[E\left[e^{\int_0^t h(u)dW(u)} \cdot e^{\int_0^s h(u)dW(u)} \mid \mathcal{F}_s^{\mathbf{W}}\right]\right]$$

Then we use the linearity properties by definition of the Wiener integral

$$= E\left[E\left[e^{\int_s^t h(u)dW(u) + \int_0^s h(u)dW(u)} \cdot e^{\int_0^s h(u)dW(u)} \mid \mathcal{F}_s^{\mathbf{W}}\right]\right]$$

$$= E\left[e^{2\int_0^s h(u)dW(u)} E\left[e^{\int_s^t h(u)dW(u)} \mid \mathcal{F}_s^{\mathbf{W}}\right]\right]$$

where we took out what is known w.r.t. $\mathcal{F}_s^{\mathbf{W}}$. But the random variable $e^{\int_s^t h(u)dW(u)}$ is a function of the increments of $\mathbf{W}$ after s and hence independent of $\mathcal{F}_s^{\mathbf{W}}$. Therefore

$$= E\left[e^{2\int_0^s h(u)dW(u)} E\left[e^{\int_s^t h(u)dW(u)}\right]\right]$$

Now we know by Collection of Formulas that $Z = \int_s^t h(u)dW(u) \in N(0, \int_s^t h^2(u)du)$. Hence we see first that

$$E\left[e^Z\right] = \psi_Z(1)$$

is the momentgenerating function $\psi_Z(t)$ of Z evaluated at $t = 1$. But both the Collection of Formulas and Beta give unanimously that

$$\psi_Z(v) = e^{\frac{\int_s^t h^2(u)du \cdot v^2}{2}}$$

(we change from t to v as the argument in order to avoid confusion with the time variable) and thus

$$E\left[e^{\int_s^t h(u)dW(u)}\right] = e^{\frac{\int_s^t h^2(u)du}{2}}.$$

But in the same manner, for $Y \in N(0, \int_0^s h^2(u)du)$.

$$E\left[e^{2\int_0^s h(u)dW(u)}\right] = \psi_Y(2) = e^{\frac{\int_s^t h^2(u)du \cdot 2^2}{2}} = e^{2\int_0^s h^2(u)du}$$

Therefore we have found for $t > s$ that

$$R_{\mathbf{Y}}(t, s) = e^{\frac{1}{2}\int_s^t h(u)du + 2\int_0^s h^2(u)du}$$

The result for $t \leq s$ is found analogously. We can write compactly this as

$$R_{\mathbf{Y}}(t, s) = e^{\frac{1}{2}\int_{min(s,t)}^{max(s,t)} h(u)du + 2\int_0^{min(t,s)} h^2(u)du}$$