Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY THURSDAY THE 9^{th} OF JANUARY 2014, 14.00 –19.00 hours.

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Thursday 9^{th} of January 2014 at 19.15.

The exam results will be announced at the latest on Friday the 24^{th} of January, 2014.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

The random variables X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 < x < y \\ 0 & \text{elsewhere.} \end{cases}$$

a) Set

$$U = Y, \quad V = \frac{X}{Y - X}.$$

Find the joint distribution of (U, V). (6 p)

b) Show that U and V are independent, and find their marginal distributions (and the names of them). (4 p)

Uppgift 2

$X \in U(-1, 1)$ and Y is symmetric Bernoulli (i.e., $\mathbf{P}(Y = 1) = \mathbf{P}(Y = -1) = \frac{1}{2}$). X and Y are independent.

a) Let $Z = X + Y$. Show that $Z \in U(-2, 2)$. (5 p)

b) Prove the standard trigonometric formula

$$\sin(2t) = 2\sin(t)\cos(t)$$

using the result in a). *Aid:* It may be helpful to consider the appropriate characteristic functions. (5 p)

Uppgift 3

Let X_1, X_2,... be independent and identically distributed with the zero-truncated Poisson distribution, i.e., their common probability mass function is with $m > 0$

$$p_X(k) = \frac{m^k}{k!} \frac{1}{e^m - 1}, \quad k = 1, 2, \ldots.$$

a) Show that the probability generating function of every X_i is

$$g_X(t) = \frac{e^{-m}}{1 - e^{-m}} \left(e^{tm} - 1 \right).$$

(5 p)

b) $N \in \text{Bin}(n, 1 - e^{-m})$, and assume that N is independent of X_1, X_2,... above. Set

$$S_N \stackrel{\text{def}}{=} X_1 + X_2 + \ldots + X_N, \quad S_N = 0, \text{ if } N = 0.$$

Show that

$$S_N \in \text{Po}(nm).$$

(5 p)

Uppgift 4

Let $U_i \in \text{Po}(1)$ be independent, for $i = 1, 2, \ldots,$. Set

$$X_n \stackrel{\text{def}}{=} U_1 + \ldots U_n.$$

a) Show that

$$\frac{X_n - n}{\sqrt{n}} \stackrel{\text{d}}{\to} N(0,1).$$

as $n \to \infty$. Please justify your steps of solution carefully. (5 p)

b) Find the limit

$$\lim_{n\to\infty} \mathbf{P}\left(X_n \leq n\right).$$

Show your calculations. (5 p)

Uppgift 5

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process, $a(t)$ and $b(t)$ are two functions such that $\int_0^1 a^2(t)dt < \infty$ and $\int_0^1 b^2(t)dt < \infty$. Let

$$Y \stackrel{\text{def}}{=} \int_0^1 a(t)dW(t), \quad Z \stackrel{\text{def}}{=} \int_0^1 b(t)dW(t),$$

a) Prove that the distribution of the random variable $(Y, Z)'$ is bivariate Gaussian. (4 p)

b) Find the parameters in the distribution of $(Y, Z)'$. (2 p)

c) Determine $E[Y \mid Z]$. (4 p)

Uppgift 6

$\{X_n\}_{n\geq 0}$ is a sequence of Gaussian variables, it can also be called a Gaussian stochastic process in discrete time.

$$\mathcal{F}_n = \sigma\{X_0, X_1, X_2, \ldots, X_{n-1}, X_n\}$$

is the sigma-field generated by the random variables $X_0, X_1, X_2, \ldots, X_{n-1}, X_n$. We take that

$$\{X_n, \mathcal{F}_n\}_{n\geq 0}$$

is a martingale.

a) Show that

$$E\left[X_{n+1}\right] = E\left[X_n\right], \quad n = 0, 1, \ldots,$$

(1 p)

b) Show that the increments $X_{n+1} - X_n$, $n = 0, 1, \ldots,$, are independent random variables. *Aid:* The result in a) can be useful in this. (6 p)

c) Set $\sigma_k^2 \stackrel{\text{def}}{=} E\left[(X_k - X_{k-1})^2\right]$, for $k = 1, 2, \ldots,$ and $\sigma_0^2 \stackrel{\text{def}}{=} \text{Var}\left[X_0\right]$. Show that

$$\text{Var}\left[X_n\right] = \sum_{k=0}^{n} \sigma_k^2.$$

Explain the steps of your solution carefully. (3 p)

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SOLUTIONS TO THE EXAM THURSDAY THE 9^{th} OF JANUARY, 2014.

Uppgift 1

a) In view of

$$U = Y, \quad V = \frac{X}{Y - X}$$

we solve

$$Y = U, X = \frac{UV}{V+1}.$$

The Jacobian determinant J of this transformation is

$$J = \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \frac{V}{V+1} & \frac{U}{(1+V)^2} \\ 1 & 0 \end{vmatrix} = -\frac{U}{(1+V)^2}.$$

Then random variables U and V have the joint density

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{uv}{v+1}, u\right) \mid J \mid$$

$$= \lambda^2 e^{-\lambda u} \frac{u}{(1+v)^2}, u > 0, v > 0.$$

Note that $y > x$ is equivalent to $u > uv/(v+1)$, i.e., $uv + u > uv$ and thus $u > 0$.

ANSWER a): $\underline{f_{U,V}(u,v) = \lambda^2 u e^{-\lambda u} \frac{1}{(1+v)^2}, u > 0, v > 0, f_{U,V}(u,v) = 0, u < 0, v < 0.}$

b) Let us write

$$f_U(u) = \lambda^2 u e^{-\lambda u}, u > 0, f_U(u) = 0, u < 0, \quad f_V(v) = \frac{1}{(1+v)^2}, v > 0, f_V(v) = 0, v < 0.$$

By the page of formulas (Appendix B) we see that $f_U(u)$ is the p.d.f. of $\Gamma(2, 1/\lambda)$, and that $f_V(v)$ is the p.d.f. of Fisher's F $F(2,2)$. Hence $f_{U,V}(u,v)$ is for all (u,v) a product of two p.d.f.s and thus U and V are independent r.v.'s.

ANSWER b): $\underline{U \in \Gamma(2, 1/\lambda), V \in F(2,2).}$

Uppgift 2

a) Since X and Y are independent we get the characteristic function of $Z = X + Y$ as

$$\varphi_Z(t) = \varphi_X(t) \cdot \varphi_Y(t).$$

Since $X \in U(-1, 1)$ we have

$$\varphi_X(t) = \frac{e^{it} - e^{-it}}{2it}.$$

By definition we get

$$\varphi_Y(t) = E\left[e^{itY}\right] = \frac{1}{2}\left(e^{it} + e^{-it}\right).$$

Thus

$$\varphi_Z = \frac{e^{it} - e^{-it}}{2it} \cdot \frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{4it}\left(e^{it} - e^{-it}\right)\left(e^{it} + e^{-it}\right)$$

and by $(x + y)(x - y) = x^2 - y^2$ we get

$$\varphi_Z(t) = \frac{1}{4it}\left(e^{2it} - e^{-2it}\right).$$

But the last expression is the characteristic function of $U(-2, 2)$ and the claim follows as asserted.

b) By the Euler formulas (Beta p. 62)

$$\varphi_Z(t) = \frac{1}{4it}\left(e^{2it} - e^{-2it}\right) = \frac{1}{2t}\sin(2t)$$

$$\varphi_Y(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos(t)$$

$$\varphi_X(t) = \frac{e^{it} - e^{-it}}{2it} = \frac{1}{t}\sin(t).$$

Hence

$$\varphi_Z(t) = \varphi_X(t) \cdot \varphi_Y(t).$$

is equivalent to the trigonometric formula

$$\sin(2t) = 2\sin(t)\cos(t)$$

as was to be proved.

Uppgift 3

a) The desired probability generating function is by definition

$$g_X(t) = \sum_{k=1}^{\infty} t^k p_X(k) = \sum_{k=1}^{\infty} t^k \frac{m^k}{k!}\frac{1}{e^m - 1} = \frac{1}{e^m - 1}\sum_{k=1}^{\infty} \frac{(tm)^k}{k!}$$

$$= \frac{1}{e^m - 1}\left(\sum_{k=0}^{\infty}\frac{(tm)^k}{k!} - 1\right) = \frac{1}{e^m - 1}\left(e^{tm} - 1\right).$$

We can write, this simplifies some effort below,

$$\frac{1}{e^m - 1} = \frac{e^{-m}}{1 - e^{-m}}$$

and thus

$$g_X(t) = \frac{e^{-m}}{1 - e^{-m}}\left(e^{tm} - 1\right).$$

b) $N \in \mathrm{Bin}(n, 1 - e^{-m})$, and is independent of X_1, X_2,... above. With

$$S_N = X_1 + X_2 + \ldots + X_N,$$

we use the composition formula to represent the probability generating function (p.g.f.) of the sum S_N as

$$g_{S_N}(t) = g_N\left(g_X(t)\right),$$

where $g_N(t)$ is the p.g.f. of N and $g_X(t)$ is the p.g.f. of X. Here

$$g_N(t) = \left(1 - (1 - e^{-m}) + t(1 - e^{-m})\right)^n = \left(e^{-m} + t(1 - e^{-m})\right)^n.$$

Then

$$g_{S_N}(t) = \left(e^{-m} + g_X(t)(1 - e^{-m})\right)^n,$$

and from part a)

$$= \left(e^{-m} + \frac{e^{-m}}{1 - e^{-m}}\left(e^{tm} - 1\right)(1 - e^{-m})\right)^n$$

$$= \left(e^{-m} + e^{-m}\left(e^{tm} - 1\right)\right)^n = \left(e^{-m} + e^{m(t-1)} - e^{-m}\right)^n$$

$$= e^{mn(t-1)}.$$

Thus we have found

$$g_{S_N}(t) = e^{mn(t-1)}$$

for all t, and this is recognized as the p.g.f of $\mathrm{Po}(nm)$. By uniqueness of p.g.f.s we have thus shown that

$$S_N \in \mathrm{Po}(nm).$$

Uppgift 4

a) Let $U_i \in \mathrm{Po}(1)$ be independent, for $i = 1, 2, \ldots,$. Set

$$X_n = U_1 + \ldots + U_n.$$

Then it follows by identical Po(1) -distributions that

$$E\left[U_i\right] = 1, \mathrm{Var}\left[U_i\right] = 1.$$

Then we get by the **central limit theorem**, that

$$\frac{X_n - n}{\sqrt{n}} = \frac{U_1 + \ldots + U_n - n}{\sqrt{n}} \stackrel{\mathrm{d}}{=\rightarrow} N(0,1),$$

as $n \to \infty$.

b) We get next that

$$P(X_n \le n) = P(X_n - n \le 0) =$$
$$= P\left(\frac{X_n - n}{\sqrt{n}} \le 0\right).$$

By definition, the convergence in distribution shown in a) means that

$$P\left(\frac{X_n - n}{\sqrt{n}} \le 0\right) = F_{\frac{X_n - n}{\sqrt{n}}}(0) \to \Phi(0) = \frac{1}{2}.$$

ANSWER b): $\underline{\lim_{n\to\infty} P(X_n \le n) = \frac{1}{2}}$.

Uppgift 5

a) Let c_1 and c_2 be two arbitrary real constants. We plan to show that the linear combination

$$c_1 Y + c_2 Z$$

is a univariate Gaussian random variable. This means that (Y, Z) is bivariate Gaussian. By construction and the properties of the Wiener integral

$$c_1 Y + c_2 Z = \int_0^1 (c_1 a(t) + c_2 b(t))\, dW(t).$$

We note that the integral is defined under the condition that

$$\int_0^1 (c_1 a(t) + c_2 b(t))^2\, dt < \infty.$$

In order to check this we note that for any real a and b

$$0 \le (a-b)^2 = a^2 - 2ab + b^2 \Leftrightarrow 2ab \le a^2 + b^2.$$

Thus

$$(a+b)^2 = a^2 + 2ab + b^2 \le 2a^2 + 2b^2,$$

and this gives

$$\int_0^1 (c_1 a(t) + c_2 b(t))^2\, dt \le 2c_1^2 \int_0^1 a^2(t)dt + 2c_2^2 \int_0^1 b^2(t)dt < \infty$$

by the stated assumptions on $a(t)$ and $b(t)$. Thus, the Wiener integral

$$\int_0^1 (c_1 a(t) + c_2 b(t))\, dW(t)$$

is well defined and is by construction a Gaussian random variable, and the desired conclusion follows.

b) By the properties of the Wiener integral in the Collection of formulas we have

$$E[Y] = E[Z] = 0$$

and

$$\mathrm{Var}[Y] = \int_0^1 a^2(t)dt, \mathrm{Var}[Z] = \int_0^1 b^2(t)dt.$$

Furthermore

$$\mathrm{Cov}(Y, Z) = E[Y \cdot Z] = \int_0^1 a(t)b(t)dt.$$

Hence we have found

$$\text{ANSWER b): } \underline{\begin{pmatrix} Y \\ Z \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \int_0^1 a^2(t)dt & \int_0^1 a(t)b(t)dt \\ \int_0^1 a(t)b(t)dt & \int_0^1 b^2(t)dt \end{pmatrix}\right).}$$

c) Since (Y, Z) is a bivariate Gaussian r.v., we have that the conditional distribution of Y given $Z = z$ is Gaussian (normal)

$$N\left(\mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_Z}(z - \mu_Z), \sigma_Y^2(1 - \rho^2)\right),$$

where $\mu_Y = E(Y)$, $\mu_Z = E(Z)$, $\sigma_Y = \sqrt{\mathrm{Var}(Y)}$, $\sigma_Z = \sqrt{\mathrm{Var}(Z)}$ and

$$\rho = \frac{\mathrm{Cov}(Z, Y)}{\sqrt{\mathrm{Var}(Z) \cdot \mathrm{Var}(Y)}}.$$

When we insert from b) we get

$$E[Y \mid Z = z] = \mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_Z}(z - \mu_Z)$$

$$= \frac{\int_0^1 a(t)b(t)dt}{\sqrt{\int_0^1 a^2(t)dt \int_0^1 b^2(t)dt}} \frac{\sqrt{\int_0^1 a^2(t)dt}}{\sqrt{\int_0^1 b^2(t)dt}} z = \frac{\int_0^1 a(t)b(t)dt}{\int_0^1 b^2(t)dt} z.$$

Hence

$$\text{ANSWER c): } \underline{E[Y \mid Z] = \frac{\int_0^1 a(t)b(t)dt}{\int_0^1 b^2(t)dt} Z.}$$

Uppgift 6

a) We use the rule of double expectation

$$E[X_{n+1}] = E[E[X_{n+1} \mid \mathcal{F}_n]] =$$

and the martingale property is $E[X_{n+1} \mid \mathcal{F}_n] = X_n$ so that

$$E[E[X_{n+1} \mid \mathcal{F}_n]] = E[X_n],$$

as was to be shown.

b) The increments $X_{n+1} - X_n$, $n = 0, 1, \ldots,$, are linear combinations Gaussian random variables and are thus Gaussian. In order to prove that Gaussian variables are independent it suffices to show that they are uncorrelated. We note first that

$$\mathrm{Cov}(X_{n+1} - X_n, X_{l+1} - X_l) = E[(X_{n+1} - X_n)(X_{l+1} - X_l)],$$

since $E[X_{n+1} - X_n] = 0$ by part a). Then

$$E[(X_{n+1} - X_n)(X_{l+1} - X_l)] = E[(X_{n+1} - X_n)X_{l+1}] - E[(X_{n+1} - X_n)X_l)]$$

Assume, without loss of generality, that $l + 1 \leq n$, i.e., $0 \leq l \leq n - 1$. Then the rule of double expectation entails that

$$E[(X_{n+1} - X_n)X_{l+1}] = E[E[(X_{n+1} - X_n)X_{l+1} \mid \mathcal{F}_n]] = E[X_{l+1}E[(X_{n+1} - X_n) \mid \mathcal{F}_n]]$$

where we took out what is known. Next by linearity of conditional expectation

$$E[(X_{n+1} - X_n) \mid \mathcal{F}_n] = E[X_{n+1} \mid \mathcal{F}_n] - E[X_n \mid \mathcal{F}_n] =$$
$$= E[X_{n+1} \mid \mathcal{F}_n] - X_n = X_n - X_n = 0,$$

where we took out what is known and invoked again the martingale property.

Similarly it follows that $E[(X_{n+1} - X_n)X_l)] = 0$ and we have shown that

$$\mathrm{Cov}(X_{n+1} - X_n, X_{l+1} - X_l) = 0$$

If $l > n$, we replace the roles of l and n in the reasoning above. Thus it follows that

$$\mathrm{Cov}(X_{n+1} - X_n, X_{l+1} - X_l) = 0$$

for $l \neq n$.

c) We have

$$X_n = (X_n - X_{n-1}) + (X_{n-1} - X_{n-2}) + \ldots + (X_1 - X_0) + X_0.$$

Since the increments are by b) independent, the standard rule for variance of a sum gives

$$\mathrm{Var}[X_n] = \mathrm{Var}[X_n - X_{n-1}] + \ldots + \mathrm{Var}[X_1 - X_0] + \mathrm{Var}[X_0].$$

Since the increments have zero means by a), we get that

$$\mathrm{Var}[X_k - X_{k-1}] = E\left[(X_k - X_{k-1})^2\right] = \sigma_k^2$$

for $k = 1, 2, \ldots,$. Thus we have shown that

$$\mathrm{Var}[X_n] = \sum_{k=0}^{n} \sigma_k^2.$$

as asserted.