Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY MONDAY 27th OCTOBER 2014, 08-13 hrs

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*Tillåtna hjälpmedel Means of assistance permitted*: Appendix B in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Monday 27th OCTOBER 2014, 2014 at 15.30.

The exam results will be announced at the latest on Tuesday the $11^{th}$ of November, 2014.

Your graded exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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**Uppgift 1**

Consider the joint probability density $f_{X,Y}(x,y)$ for the bivariate r.v. $(X,Y)$ given as

$$f_{X,Y}(x,y) = \begin{cases} 2 & x,y \geq 0, x+y \leq 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

a) Find $P(X \leq 0.9 \mid Y \leq 0.2)$. (3 p)

b) Are $X$ and $Y$ independent r.v.'s ? Justify your answer. (3 p)

c) Find $E[X \mid Y = y]$. (2 p)

d) Find $\mathrm{Var}[X \mid Y = y]$. (2 p)

**Uppgift 2**

Let $X$ be a random variable with the positive integers $1, 2, \ldots$ as values. The probability mass function of $X$ is

$$\begin{aligned} p_X(0) &= 0, \\ p_X(k) &= \int_0^1 u \cdot (1-u)^{k-1} du, \quad k = 1, 2, \ldots. \end{aligned} \tag{2}$$

Compute the probability generating function $g_X(t)$ and find it as

$$g_X(t) = 1 + \frac{(1-t)\ln(1-t)}{t}, -1 \leq t < 1.$$

(10 p)

**Uppgift 3**

a) $X$ is a continuous r.v. such that $P(X \geq 0) = 1$ and $E[X]$ is finite. Show that for any $c > 0$

$$P(X \geq c) \leq \frac{1}{c} E[X].$$

(2 p)

b) $X_1 \in U(0,1)$. We construct a sequence of r.v.s for $n \geq 2$ by

$$X_n \mid X_{n-1} = x \in U(0,x).$$

Show that

$$X_n \xrightarrow{P} 0, \quad \text{as } n \to +\infty$$

by means of the appropriate formal definition. (8 p)

## Uppgift 4

Let $\tau$ be an interoccurrence time of a Poisson process with the intensity $\lambda > 0$.

a) Show that

$$E\left[\tau - c \mid \tau \geq c\right] = \frac{1}{\lambda}$$

for any $c > 0$. (8 p)

b) What is the interpretation of the result in part a) ? (2 p)

## Uppgift 5

$\{X_i\}_{i=1}^{\infty}$ is an I.I.D. sequence of r.v.'s with

$$\mathbf{P}\left(X_i = x\right) = \begin{cases} \frac{1}{2} & x = -a \\ \frac{1}{2} & x = a, \end{cases}$$

where $a > 0$. Let $\mathbf{N} = \{N(t) | t \geq 0\}$ be a Poisson process with the intensity $\lambda > 0$. The r.v.'s $\{X_i\}_{i=1}^{\infty}$ are independent of $\mathbf{N}$. We construct a new random process $\mathbf{W} = \{W(t) | t \geq 0\}$ by the variables

$$W(t) = X_1 + X_2 + \ldots + X_{N(t)},$$
$$W(t) = 0, \qquad \text{for } N(t) = 0.$$

a) Show that the characteristic function of $W(t)$ for $t > 0$ is

$$\varphi_{W(t)}(u) = e^{\lambda t(\cos(au) - 1)}.$$

(2 p)

b) Let $a \to 0$, $\lambda \to +\infty$ in such a manner that $\lambda a^2 \to \sigma^2$. Show that then

$$W(t) \xrightarrow{d} N(0, \sigma^2 t).$$

(5 p)

c) Are the increments $W(t) - W(s)$ and $W(u) - W(v)$ independent for $t > s \geq u > v \geq 0$ ? Justify your answer concisely.
*Aid*: You do not actually need any computations. (1 p)

d) Are the increments $W(t) - W(s)$ and $W(u) - W(v)$ strictly stationary for $t > s > u > v \geq 0$ ? Justify your answer concisely.
*Aid*: You do not actually need any computations. (1 p)

e) What could the limit of $\mathbf{W} = \{W(t) | t \geq 0\}$ be as a *random process*, when $a \to 0$, $\lambda \to +\infty$ so that $\lambda a^2 \to \sigma^2$ ? Justify your answer concisely.
*Aid*: You do not need any computations. (1 p)

## Uppgift 6

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process, $a(t)$ is a function such that $\int_0^{+\infty} a^2(t)dt < \infty$. Introduce the partition of the positive real line by the points $\{t_n\}_{n=0}^{+\infty}$ so that $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots$. Let

$$X_n \stackrel{\text{def}}{=} \int_0^{t_n} a(t)dW(t), \quad n = 1, 2, \ldots, \quad X_0 = 0.$$

It holds that $\{X_n\}_{n\geq 1}$ is a Gaussian process in discrete time (You need not prove this). Let

$$\mathcal{F}_n = \sigma\left(X_0, X_1, \ldots, X_n\right)$$

be the $\sigma$-field generated by the r.v.'s $X_i$ up to time $n$.

a) Show that $\left(X_n, \mathcal{F}_n\right)_{n\geq 1}$ satisfies the martingale property. (3 p)

b) Set

$$\langle X \rangle_n \stackrel{\text{def}}{=} \int_0^{t_n} a^2(s)ds,$$

and

$$Y_n \stackrel{\text{def}}{=} X_n^2 - \langle X \rangle_n .$$

Show that $\left(Y_n, \mathcal{F}_n\right)_{n\geq 1}$ satisfies the martingale property. (5 p)

c) Express the distribution of the bivariate r.v. $\left(X_{n-1}, X_n\right)'$ in terms of the sequence of numbers $\langle X \rangle_n$. (2 p)

SOLUTIONS TO THE EXAM MONDAY THE $27^{th}$ OF October, 2014.

## Uppgift 1

a) By definition

$$P(X \leq 0.9 \mid Y \leq 0.2) = \frac{P(X \leq 0.9, Y \leq 0.2)}{P(Y \leq 0.2)}$$

To compute $P(Y \leq 0.2)$ we need the marginal p.d.f. of $Y$.

$$f_Y(y) = \int_0^{+\infty} f_{X,Y}(x,y)dx = 2\int_0^{1-y} dx = 2(1-y), \quad 0 \leq y \leq 1.$$

Thus

$$P(Y \leq 0.2) = \int_0^{0.2} f_Y(y)dy = 2\int_0^{0.2}(1-y)dy = 0.4 - \left[y^2\right]_0^{0.2}$$
$$= 0.4 - 0.04 = 0.36.$$

Furthermore, c.f. the Figure 1 below

$$P(X \leq 0.9, Y \leq 0.2) = \int_0^{0.9}\int_0^{0.2} f_{X,Y}(x,y)dydx$$
$$= 2\int_0^{0.8}\int_0^{0.2} dydx + 2\int_{0.8}^{0.9}\int_0^{1-x} dydx$$
$$= 2 \cdot 0.8 \cdot 0.2 + 2 \cdot \int_{0.8}^{0.9}(1-x)dx$$
$$= 2 \cdot 0.8 \cdot 0.2 + 2 \cdot 0.1 - 0.17 = 0.35$$

ANSWER a): $\underline{P(X \leq 0.9 \mid Y \leq 0.2) = \frac{0.35}{0.36} = 0.97.}$

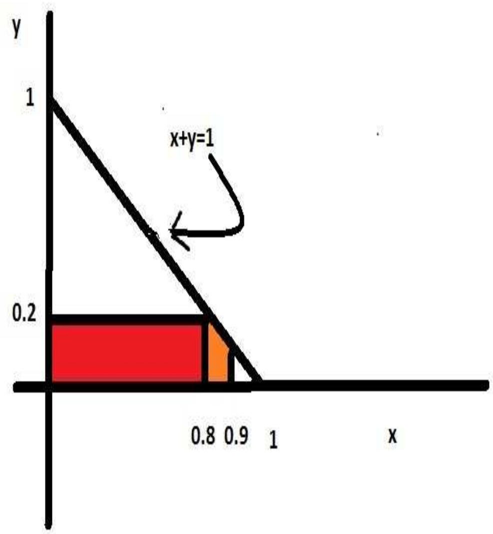
b) Are $X$ and $Y$ independent r.v.'s ? We find the marginal p.d.f. $f_X(x)$.

$$f_X(x) = \int_0^{+\infty} f_{X,Y}(x,y)dy = 2\int_0^{1-x} dy = 2(1-x), \quad 0 \leq x \leq 1.$$

It is clear that, e.g.,

$$2 = f_{X,Y}(0.5, 0.5) \neq f_X(0.5) \cdot f_Y(0.5) = 1 \cdot 1 = 1.$$

ANSWER b): $\underline{X \text{ and } Y \text{ are not independent.}}$




Figur 1: Domain of integration for $P\left(X \leq 0.9, Y \leq 0.2\right)$

c) We find the conditional p.d.f $f_{X|Y=y}(x)$.

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, \quad 0 \leq x \leq 1-y.$$

But this means that $X \mid Y = y \in U(0, 1-y)$. Hence Appendix B gives $E\left[X \mid Y = y\right] = \frac{1-y}{2}$.

ANSWER d): $\underline{E\left[X \mid Y = y\right] = \frac{1-y}{2}}$.

d) Since $X \mid Y = y \in U(0, 1-y)$, Appendix B gives $\mathrm{Var}\left[X \mid Y = y\right] = \frac{(1-y)^2}{12}$.

ANSWER d): $\underline{\mathrm{Var}\left[X \mid Y = y\right] = \frac{(1-y)^2}{12}}$.

## Uppgift 2

We start by observing that $p_X(k)$ is for $k \geq 1$ the Beta function $B(2,k)$ (Råde-Westergren section 12.5), i.e.,

$$p_X(k) = \int_0^1 u \cdot (1-u)^{k-1} du = \int_0^1 u^{2-1} \cdot (1-u)^{k-1} du = \frac{\Gamma(2)\Gamma(k)}{\Gamma(2+k)}$$

and by the property $\Gamma(k) = (k-1)!$ for any nonnegative integer $k$,

$$\frac{\Gamma(2)\Gamma(k)}{\Gamma(2+k)} = \frac{(k-1)!}{(k+1)!} = \frac{1}{(k+1)k}.$$

Now we write for $k \geq 1$

$$p_X(k) = \frac{1}{(k+1)k} = \frac{1}{k} - \frac{1}{k+1}.$$

Then we get by the general definition of the p.g.f. that

$$g_X(t) = \sum_{k=0}^{+\infty} t^k p_X(k) = \sum_{k=1}^{+\infty} t^k p_X(k) = \sum_{k=1}^{+\infty} t^k \left( \frac{1}{k} - \frac{1}{k+1} \right),$$

i.e.,

$$g_X(t) = \sum_{k=1}^{+\infty} t^k \frac{1}{k} - \sum_{k=1}^{+\infty} t^k \frac{1}{k+1}. \tag{3}$$

Collection of Formulas 13.3 (or (Råde-Westergren) gives

$$\sum_{k=1}^{+\infty} t^k \frac{1}{k} = -\ln(1-t), \quad -1 \leq t < 1.$$

Furthermore,

$$\sum_{k=1}^{+\infty} t^k \frac{1}{k+1} = \frac{1}{t} \sum_{k=1}^{+\infty} t^{k+1} \frac{1}{k+1} = \frac{1}{t} \left( \sum_{k=0}^{+\infty} t^{k+1} \frac{1}{k+1} - t \right)$$

$$= \frac{1}{t} \left( \sum_{k=1}^{+\infty} t^k \frac{1}{k} - t \right) = \frac{1}{t} \left( -\ln(1-t) - t \right).$$

Thus we insert in (3) and obtain

$$g_X(t) = -\ln(1-t) - \frac{1}{t} \left( -\ln(1-t) - t \right) = 1 + \frac{(1-t)\ln(1-t)}{t},$$

as desired.

## Uppgift 3

a) $X$ is a continuous r.v. such that $P(X \geq 0) = 1$, and its mean exists, so that

$$E[X] = \int_0^\infty x f_X(x) dx$$

$$= \underbrace{\int_0^c x f_X(x) dx}_{\geq 0} + \int_c^\infty x f_X(x) dx \geq \int_c^\infty x f_X(x) dx \geq c \int_c^\infty f_X(x) dx = c \cdot P(X \geq c),$$

which implies the stated inequality, known as Markov's inequality,

$$P(X \geq c) \leq \frac{1}{c} E[X]. \tag{4}$$

b) By definition of convergence in probability we need to show that for any $\epsilon > 0$

$$P(\mid X_n - 0 \mid > \epsilon) \to 0, \quad \text{as } n \to \infty.$$

Since $X$ is non-negative almost surely and continuous, we study

$$P(\mid X_n - 0 \mid > \epsilon) = P(X_n > \epsilon) = P(X_n \geq \epsilon).$$

The strategy is to use the inequality in a), i.e., (4). Therefore we compute

$$E[X_n].$$

By double expectation

$$E[X_n] = E[E[X_n \mid X_{n-1}]] =$$

Since $X_n \mid X_{n-1} = x \in U(0, x)$, we get

$$E[X_n \mid X_{n-1}] = \frac{1}{2} X_{n-1}.$$

Recall, if $X \in U(0, b)$, then $E[X] = \frac{b}{2}$. Thus we have found

$$E[E[X_n \mid X_{n-1}]] = \frac{1}{2} E[X_{n-1}].$$

When we iterate this

$$E[X_{n-1}] = E[E[X_{n-1} \mid X_{n-2}]] = \frac{1}{2} E[X_{n-2}],$$

so that

$$E[X_n] = \ldots = \frac{1}{2^{n-1}} E[X_1].$$

Since $X_1 \in U(0, 1)$, $E[X_1] = \frac{1}{2}$. Thus

$$E[X_n] = \frac{1}{2^n}. \tag{5}$$

The inequality in (4) yields now

$$P(X_n \geq \epsilon) \leq \frac{1}{\epsilon} E[X_n]$$

and from (5)

$$P(X_n \geq \epsilon) \leq \frac{1}{\epsilon} \frac{1}{2^n}.$$

This means clearly, as $n \to \infty$, that

$$X_n \stackrel{P}{\to} 0,$$

as was to be shown.

**Uppgift 4**

a) By the Collection of Formulas, section 2.1.3, we have the conditional density of a r.v. $X$ given the event $X \in B$ as

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P(B)} & x \in B \\ 0 & \text{elsewhere.} \end{cases}$$

We take here $X = \tau$ and $B = \{\tau \geq c\}$. We know that $\tau \in \text{Exp}\left(\frac{1}{\lambda}\right)$. Then

$$f_{\tau|\tau \geq c}(t) = \begin{cases} \frac{\lambda e^{-\lambda t}}{P(\tau \geq c)} & t \geq c \\ 0 & \text{elsewhere.} \end{cases}$$

We find, as $\tau \in \text{Exp}\left(\frac{1}{\lambda}\right)$,

$$P(\tau \geq c) = 1 - P(\tau \leq c) = 1 - \left(1 - e^{-\lambda c}\right) = e^{-\lambda c}.$$

Then

$$E\left[\tau - c \mid \tau \geq c\right] = E\left[\tau \mid \tau \geq c\right] - c. \tag{6}$$

We compute

$$E\left[\tau \mid \tau \geq c\right] = \int_{-\infty}^{\infty} t f_{\tau|\tau \geq c}(t) dt = \int_c^{\infty} t\lambda \frac{e^{-\lambda t}}{e^{-\lambda c}} dx = \frac{1}{e^{-\lambda c}} \int_c^{\infty} t\lambda e^{-\lambda t} dx.$$

Integration by parts gives

$$\int_c^{\infty} t\lambda e^{-\lambda t} dt = \left[-te^{-\lambda t}\right]_c^{\infty} + \int_c^{+\infty} e^{-\lambda t} dx = ce^{-\lambda c} + \left[-\frac{1}{\lambda} e^{-\lambda t}\right]_c^{\infty} = ce^{-\lambda c} + \frac{1}{\lambda} e^{-\lambda c}.$$

Substitution of this in (6) gives

$$E\left[\tau - c \mid \tau \geq c\right] = \frac{1}{e^{-\lambda c}} \left(ce^{-\lambda c} + \frac{1}{\lambda} e^{-\lambda c}\right) - c = c + \frac{1}{\lambda} - c = \frac{1}{\lambda},$$

as claimed.

b) This is another expression of the *lack of memory* in the exponential distribution. In some words: if $\tau \in \text{Exp}\left(\frac{1}{\lambda}\right)$, then $E[\tau] = \frac{1}{\lambda}$. Hence the result $E\left[\tau - c \mid \tau \geq c\right] = \frac{1}{\lambda}$ tells that if the next event has not arrived at time $c$, then the expectation of the time remaining till next arrival, $\tau - c$, is the same as the expectation of the total time till next arrival, or, we have gained nothing by having had to wait $c$ units of time.

## Uppgift 5

a) This is a case of a random sum of I.I.D. r.v.'s. We shall compute the characteristic function of $W(t)$, designated by $\varphi_{W(t)}(u)$, by the composition formula.

If $\{X_n\}_{n \geq 1}$ is a sequence of I.I.D. r.v.'s such that

$$\mathbf{P}\left(X_n = x\right) = \begin{cases} \frac{1}{2} & x = -a \\ \frac{1}{2} & x = a, \end{cases}$$

then the characteristic function is

$$\varphi_X(u) = E\left[e^{iuX}\right] = \frac{1}{2}e^{iua} + \frac{1}{2}e^{-iua} = \frac{1}{2}\left(e^{iua} + e^{-iua}\right) = \cos(au),$$

where we used Euler's formula for the trigonometric function cos. We know that $N(t) \in \text{Po}(\lambda t)$. By the Collection of Formulas we have the probability generating function $g_{N(t)}(u) = e^{\lambda t(u-1)}$ and we get by the composition formula

$$\begin{aligned}\varphi_{W(t)}(u) &= g_{N(t)}\left(\varphi_X(u)\right) = e^{\lambda t(\varphi_X(u)-1)} \\ &= e^{\lambda t(\cos(au)-1)}.\end{aligned}$$

i.e.,

$$\varphi_{W(t)}(u) = e^{\lambda t(\cos(au)-1)},$$

as was to be found.

b) On p. 198 (section 8.6) of L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering* we find

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{2n!} + \ldots$$

so that

$$\cos(au) - 1 = -\frac{(au)^2}{2!} + \frac{(au)^4}{4!} + O((au)^6).$$

Then we have

$$\lambda(\cos(au) - 1) = -\frac{\lambda a^2 u^2}{2!} + \frac{\lambda a^4 u^4}{4!} + O(\lambda a^6 u^6)$$

and

$$\varphi_{W(t)}(u) = e^{\lambda t(\cos(au)-1)} = e^{t\left(-\frac{\lambda a^2 u^2}{2!} + \frac{\lambda a^4 u^4}{4!} + O(\lambda a^6 u^6)\right)}.$$

Let now $a \to 0$, and $\lambda \to +\infty$ in such a manner that $\lambda a^2 \to \sigma^2$. Then we get that $\lambda a^4 = (\lambda a^2)a^2 \to \sigma^2 \cdot 0 = 0$ and the same holds for $\lambda a^{2n}$ for $n \geq 3$. Hence we get that

$$\varphi_{W(t)}(u) \to e^{-\frac{t\sigma^2 u^2}{2}}.$$

The limit function is the characteristic function of $N(0, \sigma^2 t)$. The limit function is continuous at $t = 0$. Therefore, by the uniqueness of characteristic functions we have shown that

$$W(t) \xrightarrow{d} N(0, \sigma^2 t),$$

as $a \to 0$, and $\lambda \to +\infty$ in such a manner that $\lambda a^2 \to \sigma^2$.

c) The increments $W(t) - W(s)$ and $W(u) - W(v)$ are independent for $t > s \geq u > v \geq 0$. $W(t) - W(s)$ and $W(u) - W(v)$ depend respectively on disjoint sets of independent variables $X_i$ and the corresponding independent increments of the Poisson process.

d) The increments $W(t) - W(s)$ and $W(u) - W(v)$ are strictly stationary for the reason that they depend on disjoint sets of independent variables $X_i$ and the corresponding independent increments of the Poisson process are strictly statioarny.

e) The limit of $\mathbf{W} = \{W(t) | t \geq 0\}$ should be the Wiener process, as $W(t) \xrightarrow{d} N(0, \sigma^2 t)$, and the process has independent and strictly stationay increments. We have not, however, shown that the increments are Gaussian.

## Uppgift 6

a) We need to show that

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n \quad \text{(a.s.)}$$

We have

$$E[X_{n+1} \mid \mathcal{F}_n] = E\left[\int_0^{t_{n+1}} a(t)dW(t) \mid \mathcal{F}_n\right]$$

$$= E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) + \int_0^{t_n} a(t)dW(t) \mid \mathcal{F}_n\right]$$

$$= E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) \mid \mathcal{F}_n\right] + E\left[\int_0^{t_n} a(t)dW(t) \mid \mathcal{F}_n\right]$$

$$= E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) \mid \mathcal{F}_n\right] + E[X_n \mid \mathcal{F}_n]$$

but we take out what is known, since $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, and get

$$= E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) \mid \mathcal{F}_n\right] + X_n$$

and since an independent condition (by properties of the Wiener process) drops out and by a property of the Wiener integral

$$= \underbrace{E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t)\right]}_{=0} + X_n = X_n.$$

Hence we have established the martingale property.

b) Again, we study

$$E[Y_{n+1} \mid \mathcal{F}_n] = E\left[X_{n+1}^2 - \langle X\rangle_{n+1} \mid \mathcal{F}_n\right]$$

$$= E\left[X_{n+1}^2 \mid \mathcal{F}_n\right] - \int_0^{t_{n+1}} a^2(t)dt \tag{7}$$

since $E\left[\langle X\rangle_{n+1} \mid \mathcal{F}_n\right] = E\left[\int_0^{t_{n+1}} a^2(t)dt \mid \mathcal{F}_n\right] = \int_0^{t_{n+1}} a^2(t)dt$. Then

$$E\left[X_{n+1}^2 \mid \mathcal{F}_n\right] = E\left[\left(\int_{t_n}^{t_{n+1}} a(t)dW(t) + \int_0^{t_n} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right]$$

$$= E\left[\left(\int_{t_n}^{t_{n+1}} a(t)dW(t)\right)^2 + 2\int_{t_n}^{t_{n+1}} a(t)dW(t)\int_0^{t_n} a(t)dW(t) + \left(\int_0^{t_n} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right]$$

$$= E\left[\left(\int_{t_n}^{t_{n+1}} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right]$$

$$+2E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t)\int_0^{t_n} a(t)dW(t) \mid \mathcal{F}_n\right]$$

$$+E\left[\left(\int_0^{t_n} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right].$$

Here an independent condition drops out and a property of the Wiener integral yields

$$E\left[\left(\int_{t_n}^{t_{n+1}} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right] = E\left[\left(\int_{t_n}^{t_{n+1}} a(t)dW(t)\right)^2\right] = \int_{t_n}^{t_{n+1}} a^2(t)dt.$$

We take out what is known

$$E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) \int_0^{t_n} a(t)dW(t) \mid \mathcal{F}_n\right] = E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) X_n \mid \mathcal{F}_n\right]$$

$$= X_n E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t) \mid \mathcal{F}_n\right] = X_n \cdot E\left[\int_{t_n}^{t_{n+1}} a(t)dW(t)\right] = X_n \cdot 0 = 0,$$

where again an indendent condition dropped out, and we used a property of the Wiener integral. Finally

$$E\left[\left(\int_0^{t_n} a(t)dW(t)\right)^2 \mid \mathcal{F}_n\right] = E\left[X_n^2 \mid \mathcal{F}_n\right] = X_n^2,$$

where we took out what is known. Taking into account (7) we have found

$$E\left[Y_{n+1} \mid \mathcal{F}_n\right] = \int_{t_n}^{t_{n+1}} a^2(t)dt + X_n^2 - \int_0^{t_{n+1}} a^2(t)dt = X_n^2 - \int_0^{t_n} a^2(t)dt = Y_n,$$

as was to be proved.

> We have found a decomposition $X_n^2 = Y_n + \langle X \rangle_n$, where $\langle X \rangle_n < \langle X \rangle_{n+1} < \ldots$ is increasing. In martingale theory the sequence $\langle X \rangle_n$ is called the *compensator* of $X_n^2$ (which is a submartingale).

c) The bivariate r.v. $(X_{n-1}, X_n)'$ is bivariate Gaussian. The means are zero by Wiener integrals, as for all $n$

$$E\left[X_n\right] = E\left[\int_0^{t_n} a(t)dW(t)\right] = 0.$$

For all $n$

$$\mathrm{Var}\left[X_n\right] = \int_0^{t_n} a^2(t)dt = \langle X \rangle_n .$$

The covariance is, as means are zero, by a property of the Wiener integral,

$$\mathrm{Cov}\left(X_{n-1}, X_n\right) = E\left[X_{n-1} \cdot X_n\right] = \int_0^{\min(t_{n-1},t_n)} a^2(t)dt = \int_0^{t_{n-1}} a^2(t)dt = \langle X \rangle_{n-1} .$$

Hence we have

ANSWER c): $\underline{\begin{pmatrix} X_{n-1} \\ X_n \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \langle X \rangle_{n-1} & \langle X \rangle_{n-1} \\ \langle X \rangle_{n-1} & \langle X \rangle_n \end{pmatrix}\right)}.$