Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY 7th JANUARY 2015, 14.00-19.00 hrs

Examinator: Timo Koski, tel. 070 2370047, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix B in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 7th January 2015, 2015 at 21.30.

The exam results will be announced at the latest on Friday the 23^{rd} of January, 2015.

Your graded exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

Lycka till!

- - - - - - - - - - - - - - - - - - - -

Uppgift 1

Consider the joint probability density $f_{X,Y}(x,y)$ for the bivariate r.v. (X,Y) given as

$$f_{X,Y}(x,y) = \begin{cases} 2 & x, y \geq 0, x + y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

a) Find $E[X]$ and $E[Y]$. (2 p)

b) Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$. (2 p)

c) Find $\mathrm{Cov}(X,Y)$. (3 p)

d) Find the coefficient of correlation $\rho_{X,Y}$. (1 p)

e) How does the value of $\rho_{X,Y}$ reflect the properties of $f_{X,Y}(x,y)$? (2 p)

Uppgift 2

$N \in \mathrm{Po}(\lambda)$, $\lambda > 0$ and $X \mid N = n \in \mathrm{Bin}(n,p)$.

a) Show that $E[X] = \lambda \cdot p$ and $\mathrm{Var}[X] = \lambda \cdot p$. (2 p)

b) Find the distribution of X. Countercheck your result against the findings in a). (8 p)

Uppgift 3

A random variable X has the p.d.f., $\sigma > 0$,

$$f_X(x) = \begin{cases} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Then we say that $X \in HN(0,\sigma^2)$, and that X is a half-normal (or folded normal) distributed random variable.

a) Let $X \in HN(0,\sigma^2)$ and $c > 0$ be a constant. What is the distribution of $\frac{X}{c}$? (2 p)

b) Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent $HN(0,\sigma^2)$ -distributed random variables. Find the limiting distribution of

$$\frac{X_1}{\frac{1}{n}\sum_{i=1}^{n} X_i},$$

as $n \to +\infty$. Justify your solution. (8 p)

Uppgift 4

$\mathbf{X} = \{X(t) \mid -\infty < t < +\infty\}$ is a Gaussian stochastic process with the mean function $\mu_{\mathbf{X}}(t) = 0$ for all t, and the autocorrelation function

$$R_{\mathbf{X}}(h) = E\left[X(t)X(s)\right] = e^{-|h|}, \quad h = t - s.$$

Let

$$v_1 \stackrel{\text{def}}{=} X(1) - E\left[X(1) \mid X(0)\right]$$

and

$$v_2 \stackrel{\text{def}}{=} X(2) - E\left[X(2) \mid X(0)\right].$$

a) Show that $(v_1, v_2)'$ is a bivariate Gaussian random variable. (4 p)

b) Find the mean vector and covariance matrix of $(v_1, v_2)'$. (5 p)

c) $h > 0$. What is the distribution of $(v_{1,h}, v_{2,h})'$, when

$$v_{1,h} \stackrel{\text{def}}{=} X(1+h) - E\left[X(1+h) \mid X(h)\right]$$

and

$$v_{2,h} \stackrel{\text{def}}{=} X(2+h) - E\left[X(2+h) \mid X(h)\right].$$

You do not need any calculations. (1 p)

Uppgift 5

$\{X_i\}_{i=1}^{\infty}$ is an I.I.D. sequence of r.v.'s with

$$\mathbf{P}\left(X_i = x\right) = \begin{cases} \frac{1}{2} & x = -a \\ \frac{1}{2} & x = a, \end{cases}$$

where $a > 0$. Let $\mathbf{N} = \{N(t) | t \geq 0\}$ be a Poisson process with the intensity $\lambda > 0$. The r.v.'s $\{X_i\}_{i=1}^{\infty}$ are independent of $\mathbf{N}$. We construct a new random process $\mathbf{W} = \{W(t) | t \geq 0\}$ by the variables

$$W(t) = X_1 + X_2 + \ldots + X_{N(t)},$$

$$W(t) = 0, \quad \text{for } N(t) = 0.$$

a) Show that the characteristic function of $W(t) - W(s)$ for $t > s$ is

$$\varphi_{W(t)-W(s)}(u) = e^{\lambda(t-s)(\cos(au)-1)}.$$

Show your calculations. *Aid:* To condition on the values of $N(t) - N(s)$ may be helpful. (7 p)

b) Prove that the increments $W(t) - W(s)$ are strictly stationary for $t > s$. (2 p)

c) Are the increments $W(t) - W(s)$ and $W(u) - W(v)$ independent for $t > s \geq u > v \geq 0$? Justify your answer concisely. *Aid*: You do not actually need any computations. (1 p)

Uppgift 6

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process, $a(t)$ is a function such that $\int_0^{+\infty} a^2(t)dt < \infty$. Introduce the partition of the positive real line by the points $\{t_n\}_{n=0}^{+\infty}$ so that $0 = t_0 < t_1 < t_2 < \ldots < t_n < \ldots$. Let

$$X_n \stackrel{\text{def}}{=} \int_0^{t_n} a(t)dW(t), \quad n = 1, 2, \ldots, \quad X_0 = 0.$$

It holds that $\{X_n\}_{n\geq 1}$ is a Gaussian process in discrete time (You need not prove this).

a) Define

$$Y_n \stackrel{\text{def}}{=} X_n - X_{n-1}, \quad Y_0 = 0.$$

Show that $\{Y_n\}_{n=1}^{+\infty}$ is a sequence of independent r.v.'s. You may need in this

$$\mathcal{F}_n = \sigma\left(X_0, X_1, \ldots, X_n\right),$$

which is the σ-field generated by the r.v.'s X_i up to time n.

(6 p)

b) Set $\sigma_k^2 \stackrel{\text{def}}{=} \text{Var}[Y_k]$. Show that

$$\text{Var}[X_n] = \sum_{k=1}^{n} \sigma_k^2.$$

Does this result agree with the formula for the variance of $\int_0^{t_n} a(t)dW(t)$? (4 p)

SOLUTIONS TO THE EXAM WEDNESDAY THE 7^{th} OF JANUARY, 2015.

Uppgift 1

a) We find the marginal p.d.f. $f_X(x)$

$$f_X(x) = \int_0^{+\infty} f_{X,Y}(x,y)dy = 2\int_0^{1-x} dy = 2(1-x), \quad 0 \leq x \leq 1,$$

and the marginal p.d.f. $f_Y(y)$,

$$f_Y(y) = \int_0^{+\infty} f_{X,Y}(x,y)dx = 2\int_0^{1-y} dx = 2(1-y), \quad 0 \leq y \leq 1.$$

Then

$$E[X] = \int_0^1 x \cdot 2(1-x)dx = 2\left(\int_0^1 xdx - \int_0^1 x^2dx\right)$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

Clearly $E[Y] = E[X]$.

ANSWER a): $\underline{E[Y] = E[X] = \frac{1}{3}.}$

b) We use Steiner's formula

$$\text{Var}[X] = E[X^2] - (E[X])^2.$$

There, by the law of the unconscious statistician,

$$E[X^2] = \int_0^1 x^2 \cdot 2(1-x)dx = 2\int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}.$$

Thus

$$\text{Var}[X] = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

Clearly $\text{Var}[Y] = \text{Var}[X]$.

ANSWER b): $\underline{\text{Var}[Y] = \text{Var}[X] = \frac{1}{18}.}$

c) The covariance $\mathrm{Cov}\,(X,Y)$ is conveniently computed as

$$\mathrm{Cov}\,(X,Y) = E\left[XY\right] - E[X]E[Y].$$

We have

$$E\left[XY\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy = 2\int_0^1 x\int_0^{1-x} ydydx =$$

$$= 2\int_0^1 x\frac{(1-x)^2}{2}dx = \int_0^1 (x-2x^2+x^3)dx$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{6-8+3}{12} = \frac{1}{12}.$$

Thus, by the preceding and by part a)

$$\mathrm{Cov}\,(X,Y) = \frac{1}{12} - \frac{1}{9} = \frac{3-4}{36} = -\frac{1}{36}.$$

ANSWER c): $\underline{\mathrm{Cov}\,(X,Y) = -\frac{1}{36}}$.

d) The coefficient of correlation $\rho_{X,Y}$ is by the collection of formulas

$$\rho_{X,Y} = \frac{\mathrm{Cov}\,(X,Y)}{\sqrt{\mathrm{Var}\left[Y\right]\cdot \mathrm{Var}\left[X\right]}}.$$

When one inserts from the above

$$\rho_{X,Y} = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2}.$$

ANSWER d): $\underline{\rho_{X,Y} = -\frac{1}{2}}$.

e) How does the value of $\rho_{X,Y}$ reflect the properties of $f_{X,Y}(x,y)$? The covariance $\mathrm{Cov}\,(X,Y)$ is defined as

$$\mathrm{Cov}\,(X,Y) = E\left[(X-E[X])(Y-E[Y])\right].$$

Hence we are looking here how the product

$$(X-\frac{1}{3})(Y-\frac{1}{3})$$

varies in the average. Suppose the outcome $X=x$ is such that $x-\frac{1}{3}<0$, i.e., $x<\frac{1}{3}$. We know that

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}, \quad 0\leq y\leq 1-x.$$

This means that $Y \mid X=x \in U(0,1-x)$. We find that

$$P\left(Y<1/3 \mid X=x\right) = \frac{1/3}{1-x}.$$

Since $x < \frac{1}{3}$, we get that $1 - x > 1 - \frac{1}{3} = \frac{2}{3}$, so that

$$\frac{1/3}{1-x} < \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}.$$

Hence we see that, if $x < \frac{1}{3}$, then

$$P\left(Y < 1/3 \mid X = x\right) < \frac{1}{2}.$$

i.e, if $x - \frac{1}{3} < 0$, then $Y - \frac{1}{3}$ is more likely to be positive than negative. Similary, one sees that if $x - \frac{1}{3} > 0$, then $Y - \frac{1}{3}$ is more likely to negative than positive. Hence the sign of $(X - \frac{1}{3})(Y - \frac{1}{3})$ would, in the average, be negative. Thus it seems natural that $\rho_{X,Y} < 0$.

A passable solution to e) need not be as detailed as the one above. One can also argue qualitatively from the property that if $X = x$, then $0 < Y \leq 1 - x$. Thus if x is small, then there is a larger probability for Y to become large.

Uppgift 2

a) We have by double expectation and Doob-Dynkin

$$E[X] = E[E[X|N]] = E[Np] = pE[N] = \lambda \cdot p,$$

since $X \mid N = n \in \text{Bin}(n, p)$ and $E\left[X \mid N = n\right] = np$ and $E[N] = \lambda$.

Next, by the collection of formulas, section 2.4.,

$$\text{Var}[X] = \text{Var}[E[X|N]] + E[\text{Var}[X|N]].$$

Since $E[X|N] = Np$, as found above, and $\text{Var}[X|N] = Np(1-p)$, as $X \mid N = n \in \text{Bin}(n, p)$, and $\text{Var}[N] = \lambda$, we get

$$\text{Var}[X] = p^2\lambda + p(1-p)\lambda = p\lambda,$$

as was claimed.

b) In order to find the distribution of X, we compute the p.g.f. of X, $g_X(t)$, as X is a non-negative integervalued r.v.. By definition and double expectation

$$g_X(t) = E[t^X] = E[E[t^X|N]] = \sum_{n=0}^{+\infty}[E[t^X|N = n]p_N(n).$$

We have by collection of formulas (section 8)

$$E[t^X|N = n] = ((1-p) + pt)^n,$$

since $X \mid N = n \in \text{Bin}(n, p)$. Then, since $N \in \text{Po}(\lambda)$,

$$\sum_{n=0}^{+\infty}[E[t^X|N = n]p_N(n) = \sum_{n=0}^{+\infty}((1-p) + pt)^n e^{-\lambda}\frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{+\infty} \frac{(((1-p)+pt)\lambda)^n}{n!}$$

$$= e^{-\lambda} e^{((1-p)+pt)\lambda} = e^{p\lambda(t-1)}.$$

Thus we have found that

$$g_X(t) = e^{p\lambda(t-1)},$$

and section 8 in the collection of formulas identifies this function as the p.g.f. of $\text{Po}(p\lambda)$. Hence uniqueness of p.g.f. yields that $X \in \text{Po}(\lambda p)$ and the property $\text{Var}[X] = E[X] = p\lambda$ holds.

ANSWER b): $\underline{X \in \text{Po}(\lambda p).}$

Uppgift 3

a) We compute the distribution function $F_{\frac{X}{c}}(x)$ of $\frac{X}{c}$. We assume $x > 0$. By definition

$$F_{\frac{X}{c}}(x) = P\left(\frac{X}{c} \leq x\right) = P\left(X \leq cx\right)$$

since $c > 0$. As $X \in HN(0, \sigma^2)$,

$$P\left(X \leq cx\right) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_0^{cx} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt.$$

We differentiate this w.r.t. x to find the p.d.f.

$$f_{\frac{X}{c}}(x) = \frac{d}{dx} F_{\frac{X}{c}}(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{(cx)^2}{2\sigma^2}\right) \cdot c = \frac{\sqrt{2}}{(\sigma/c)\sqrt{\pi}} \exp\left(-\frac{x^2}{2(\sigma/c)^2}\right).$$

But the rightmost expression in the above is recognized as the p.d.f. of $HN(0, (\sigma/c)^2)$.

ANSWER a): $\underline{X \in HN(0, (\sigma/c)^2).}$

b) As $X_1, \ldots, X_n, \ldots$ is a sequence of independent $HN(0, \sigma^2)$ -distributed random variables, we get by the (Weak) Law of Large Numbers that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \stackrel{P}{\to} E[X_1],$$

as $n \to +\infty$. We have

$$E[X_1] = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_0^{+\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx,$$

an expression clearly equaling

$$= \frac{\sigma^2\sqrt{2}}{\sigma\sqrt{\pi}} \left[-\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_0^{+\infty}$$

$$= \frac{\sigma\sqrt{2}}{\sqrt{\pi}}.$$

Thus we find by the Cramér-Slutsky theorem that, as $n \to +\infty$,

$$\frac{X_1}{\frac{1}{n}\sum_{i=1}^{n} X_i} \xrightarrow{\text{d}} \frac{X_1}{\frac{\sigma\sqrt{2}}{\sqrt{\pi}}}.$$

(Note that, of course, $X_1 \xrightarrow{\text{d}} X_1$). We take in a) $c = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}$, and since $X_1 \in HN(0, \sigma^2)$, we get by part a) that

$$\frac{X_1}{\frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \in HN(0, \frac{\pi}{2}).$$

ANSWER b): $\underline{\frac{X_1}{\frac{1}{n}\sum_{i=1}^{n} X_i} \xrightarrow{\text{d}} HN(0, \frac{\pi}{2}) \quad \text{as} \quad n \to \infty.}$

Uppgift 4

a) $\mathbf{X} = \{X(t) \mid -\infty < t < +\infty\}$ is a Gaussian stochastic process with the constant mean function $\mu_{\mathbf{X}}(t) = 0$ for all t, and the autocorrelation function

$$R_{\mathbf{X}}(h) = E\left[X(t)X(s)\right] = e^{-|h|}, \quad h = t - s.$$

Hence the process is weakly stationary, and thus as it is a Gaussian process, strictly stationary, too. Then we get that

$$E[X(0)] = E[X(1)] = E[X(2)] = 0.$$

and

$$\text{Var}[X(0)] = \text{Var}[X(1)] = \text{Var}[X(2)] = R_{\mathbf{X}}(0) = 1.$$

In addition, the coefficient of correlation is

$$\rho_{\mathbf{X}}(h) = \frac{\text{Cov}(X(t+h), X(t))}{\sqrt{\text{Var}[X(t+h)] \cdot \text{Var}[X(t)]}} = R_{\mathbf{X}}(h).$$

By the Collection of Formulas, section 9.2., we have that

$$E\left[X(1) \mid X(0)\right] = \rho_{\mathbf{X}}(1)X(0)$$

and

$$E\left[X(2) \mid X(0)\right] = \rho_{\mathbf{X}}(2)X(0).$$

Then

$$v_1 = X(1) - E\left[X(1) \mid X(0)\right] = X(1) - \rho_{\mathbf{X}}(1)X(0)$$

and

$$v_2 = X(2) - E\left[X(2) \mid X(0)\right] = X(2) - \rho_{\mathbf{X}}(2)X(0).$$

Thus we can write

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -e^{-1} \\ 1 & 0 & -e^{-2} \end{pmatrix} \begin{pmatrix} X(2) \\ X(1) \\ X(0) \end{pmatrix}.$$

This shows $(v_1, v_2)'$ as a linear transformation of a multivariate (trivariate) Gaussian vector. Therefore $(v_1, v_2)'$ is bivariate Gaussian.

b) It is clear that $(0,0)'$ is the mean vector of $(v_1, v_2)'$. We find the covariance matrix. The diagonal elements are the variances. A rule from the Collection of Formulas gives

$$\text{Var}[v_1] = \text{Var}[X(1)] + \rho_{\mathbf{X}}(1)^2 \text{Var}[X(0)] - 2\rho_{\mathbf{X}}(1) R_{\mathbf{X}}(1) = 1 + e^{-2} - 2e^{-2} = 1 - e^{-2}.$$

Furthermore,

$$\text{Var}[v_2] = \text{Var}[X(2)] + \rho_{\mathbf{X}}(2)^2 \text{Var}[X(0)] - 2\rho_{\mathbf{X}}(2) R_{\mathbf{X}}(2) = 1 + e^{-4} - 2e^{-4} = 1 - e^{-4}.$$

We need the covariances.

$$E[v_1 v_2] = E\left[(X(1) - \rho_{\mathbf{X}}(1)X(0))(X(2) - \rho_{\mathbf{X}}(2)X(0))\right]$$

$$= E[X(1)X(2)] - \rho_{\mathbf{X}}(2)E[X(1)X(0)] - \rho_{\mathbf{X}}(1)E[X(0)X(2)] + \rho_{\mathbf{X}}(1)\rho_{\mathbf{X}}(2)E\left[X^2(0)\right]$$

$$= \rho_{\mathbf{X}}(1) - \rho_{\mathbf{X}}(2)\rho_{\mathbf{X}}(1) - \rho_{\mathbf{X}}(1)\rho_{\mathbf{X}}(2) + \rho_{\mathbf{X}}(1)\rho_{\mathbf{X}}(2)$$

$$= \rho_{\mathbf{X}}(1) - \rho_{\mathbf{X}}(2)\rho_{\mathbf{X}}(1) = e^{-1}\left(1 - e^{-2}\right)$$

Thus we have

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 - e^{-2} & e^{-1}(1 - e^{-2}) \\ e^{-1}(1 - e^{-2}) & 1 - e^{-4} \end{pmatrix} \right).$$

ANSWER b): $\underline{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 - e^{-2} & e^{-1}(1 - e^{-2}) \\ e^{-1}(1 - e^{-2}) & 1 - e^{-4} \end{pmatrix} \right).}$

c) Since the process $\mathbf{X}$ is strictly stationary,

$$\begin{pmatrix} X(2+h) \\ X(1+h) \\ X(h) \end{pmatrix} \stackrel{\text{d}}{=} \begin{pmatrix} X(2) \\ X(1) \\ X(0) \end{pmatrix}.$$

Since

$$\begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -e^{-1} \\ 1 & 0 & -e^{-2} \end{pmatrix} \begin{pmatrix} X(2+h) \\ X(1+h) \\ X(h) \end{pmatrix}$$

it follows by the preceding that $(v_{1,h}, v_{2,h})' \stackrel{d}{=} (v_1, v_2)'$.

ANSWER b): $\underline{(v_{1,h}, v_{2,h})' \stackrel{d}{=} (v_1, v_2)'.}$

Uppgift 5

a) The characteristic function of $W(t) - W(s)$, designated by $\varphi_{W(t)-W(s)}(u)$, is by definition

$$\varphi_{W(t)-W(s)}(u) = E\left[e^{iuW(t)-W(s)}\right]$$

We apply double expectation by the trick

$$E\left[e^{iuW(t)-W(s)}\right] = E\left[E\left[e^{iuW(t)-W(s)} | N(t) - N(s)\right]\right].$$

This equals

$$E\left[E\left[e^{iuW(t)-W(s)}|N(t)-N(s)\right]\right] = \sum_{k=0}^{+\infty} E\left[e^{iuW(t)-W(s)}|N(t)-N(s)=k\right] p_{N(t)-N(s)}(k).$$

We have by construction, as $t > s$ that

$$W(t) - W(s) = X_{N(s)+1} + \ldots + X_{N(t)}.$$

Given that $N(t)-N(s) = k$, we have k jumps in $(s,t]$ and thus $X_{N(s)+1}+\ldots+X_{N(t)}$ is a sum of k independent random variables that are independent of the Poisson process **N**. Thus

$$E\left[e^{iuW(t)-W(s)}|N(t)-N(s)=k\right] = E\left[(\varphi_X(u))^k\right],$$

where $\varphi_X(u)$ is the characteristic function of an r.v. such that

$$\mathbf{P}(X = x) = \begin{cases} \frac{1}{2} & x = -a \\ \frac{1}{2} & x = a. \end{cases}$$

Thus

$$\varphi_X(u) = E\left[e^{iuX}\right] = \frac{1}{2}e^{iua} + \frac{1}{2}e^{-iua} = \frac{1}{2}\left(e^{iua} + e^{-iua}\right) = \cos(au),$$

where we used Euler's formula for the trigonometric function cos. We insert in the preceding to get

$$\sum_{k=0}^{+\infty} E\left[e^{iuW(t)-W(s)}|N(t)-N(s)=k\right] p_{N(t)-N(s)}(k) = \sum_{k=0}^{+\infty} \cos(au)^k p_{N(t)-N(s)}(k).$$

We know that $N(t) - N(s) \in \text{Po}(\lambda(t-s))$ and thus

$$p_{N(t)-N(s)}(k) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^k}{k!}.$$

A substitution of this gives

$$\sum_{k=0}^{+\infty} \cos(au)^k p_{N(t)-N(s)}(k) = e^{-\lambda(t-s)}\sum_{k=0}^{+\infty}\frac{(\cos(au)\lambda(t-s))^k}{k!}.$$

Collecting from the preceding we have established

$$E\left[e^{iuW(t)-W(s)}\right] = e^{-\lambda(t-s)}e^{\cos(au)\lambda(t-s)} = e^{\lambda(t-s)(\cos(au)-1)},$$

as was claimed.

Alternative argument

$$W(t) - W(s) = X_{N(s)+1} + \ldots + X_{N(t)} \stackrel{d}{=} \sum_{i=1}^{N(t)-N(s)} X_i$$

Then the result follows by an application of the composition formula.

b) Let us consider the joint distribution of the increments

$$\{W(t_i) - W(t_{i-1})\}_{i=1}^{n}, \quad t_0 < t_1 < \ldots < t_i < t_{i+2} < \ldots$$

Since the characteristic function of an individual increment depends, by the case a) above, only on the difference of the time arguments, we see that for any i and any h

$$W(t_i + h) - W(t_{i-1} + h) \stackrel{d}{=} W(t_i) - W(t_{i-1}).$$

The increments are independent as they depend, respectively, on the corresponding independent increments of the Poisson process and on the independent random variables X_i. Hence

$$\{W(t_i + h) - W(t_{i-1} + h)\}_{i=1}^{n} \stackrel{d}{=} \{W(t_i) - W(t_{i-1})\}_{i=1}^{n},$$

and the increments are seen to be strictly stationary.

c) The increments $W(t) - W(s)$ and $W(u) - W(v)$ are independent for $t > s \geq u > v \geq 0$. $W(t) - W(s)$ and $W(u) - W(v)$ depend respectively on the corresponding independent increments of the Poisson process and on the independent random variables X_i.

Uppgift 6

a) The sequence

$$Y_n = X_n - X_{n-1}, \quad Y_0 = 0.$$

is a sequence of Gaussian r.v.'s. Hence, to verify that $\{Y_n\}_{n=1}^{+\infty}$ are independent r.v.'s., it suffices to show that $\{Y_n\}_{n=1}^{+\infty}$ are uncorrelated, or that

$$\text{Cov}(Y_n, Y_k) = 0$$

for $k \neq n$. We have that

$$\text{Cov}(Y_n, Y_k) = E[Y_n Y_k] - E[Y_n]E[Y_k].$$

We see immediately that for all n

$$E[Y_n] = E[\int_0^{t_n} a(t)dW(t)] - E[\int_0^{t_{n-1}} a(t)dW(t)] = 0 - 0 = 0$$

by a property of the Wiener integrals. Thus we need to show that for $n \neq k$

$$E[Y_n Y_k] = 0$$

There are at least two different proofs (A & B) of this. We present these both, but it is not required to give two proofs in the exam.

Proof A: We have by construction

$$Y_n = X_n - X_{n-1} = \int_{t_{n-1}}^{t_n} a(t)dW(t)$$

We assume that $n > k$ and compute

$$E[Y_n Y_k] = E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t) \cdot \int_{t_{k-1}}^{t_k} a(t)dW(t)\right]$$

Let us define

$$f(t) \stackrel{\text{def}}{=} \begin{cases} a(t), & \text{if } t_{n-1} < t \le t_n \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$g(t) \stackrel{\text{def}}{=} \begin{cases} a(t), & \text{if } t_{k-1} < t \le t_k \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\int_{t_{n-1}}^{t_n} a(t)dW(t) = \int_0^{t_n} f(t)dW(t), \int_{t_{k-1}}^{t_k} a(t)dW(t) = \int_0^{t_n} g(t)dW(t)$$

Thus a result in section 10 in the Collection of Formulas gives now

$$E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t) \cdot \int_{t_{k-1}}^{t_k} a(t)dW(t)\right] = E\left[\int_0^{t_n} f(t)dW(t) \cdot \int_0^{t_n} g(t)dW(t)\right]$$

$$= \int_0^{t_n} f(t)g(t)dt = 0$$

by construction of $f(t)$ and $g(t)$. We may consider $k > n$ and get the same result. Thus it follows that

$$E[Y_n Y_k] = 0$$

for $k \neq n$.

Proof B:

$$\mathcal{F}_n = \sigma\left(X_0, X_1, \ldots, X_n\right),$$

is the σ-field generated by the r.v.'s X_i up to time n.

$$E[Y_n Y_k] = E\left[(X_n - X_{n-1})(X_k - X_{k-1})\right] = E\left[(X_n - X_{n-1})X_k\right] - E\left[(X_n - X_{n-1})X_{k-1}\right]$$

Assume, without loss of generality, that $k < n$, i.e., $0 \le k \le n-1$. Then the rule of double expectation entails that

$$E\left[(X_n - X_{n-1})X_k\right] = E\left[E\left[(X_n - X_{n-1})X_k \mid \mathcal{F}_{n-1}\right]\right] = E\left[X_k E\left[(X_n - X_{n-1}) \mid \mathcal{F}_{n-1}\right]\right]$$

where we took out what is known. Next by linearity of conditional expectation

$$E\left[(X_n - X_{n-1}) \mid \mathcal{F}_{n-1}\right] = E\left[X_n \mid \mathcal{F}_{n-1}\right] - E\left[X_{n-1} \mid \mathcal{F}_{n-1}\right] =$$

$$= E\left[X_n \mid \mathcal{F}_{n-1}\right] - X_{n-1} = E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t) + X_{n-1} \mid \mathcal{F}_{n-1}\right] - X_{n-1}$$

$$= E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t) \mid \mathcal{F}_{n-1}\right] + X_{n-1} - X_{n-1},$$

where we took out what is known. Next we observe

$$= E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t) \mid \mathcal{F}_{n-1}\right] = E\left[\int_{t_{n-1}}^{t_n} a(t)dW(t)\right] = 0,$$

where an independent condition dropped out and a property of the Wiener integral was applied.

It follows similarly that $E\left[(X_n - X_{n-1})X_{k-1})\right] = 0$ and we have shown that

$$\text{Cov}(X_n - X_{n-1}, X_k - X_{k-1}) = 0$$

If $k > n$, we replace the roles of k and n in the reasoning above. Thus it follows that

$$\text{Cov}(X_n - X_{n-1}, X_k - X_{k-1}) = 0$$

for $k \neq n$.

b) We have

$$X_n = (X_n - X_{n-1}) + (X_{n-1} - X_{n-2}) + \ldots + (X_1 - X_0) + X_0.$$

Since the increments are by a) independent, the standard rule for variance of a sum gives

$$\text{Var}\left[X_n\right] = \text{Var}\left[X_n - X_{n-1}\right] + \ldots + \text{Var}\left[X_1 - X_0\right] + \text{Var}\left[X_0\right].$$

We have

$$\text{Var}\left[X_k - X_{k-1}\right] = \text{Var}[Y_k] = \sigma_k^2$$

for $k = 1, 2, \ldots,$. Thus we have shown that

$$\text{Var}\left[X_n\right] = \sum_{k=1}^{n} \sigma_k^2.$$

as asserted.

Does this result agree with the formula for the variance of $\int_0^{t_n} a(t)dW(t)$? Yes, since

$$\sigma_k^2 = \int_{t_{k-1}}^{t_k} a^2(t)dt$$

and since $\{t_n\}_{n\geq 0}$ is a partition, we have

$$\sum_{k=1}^{n} \sigma_k^2 = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} a^2(t)dt = \int_0^{t_n} a^2(t)dt = \text{Var}[\int_0^{t_n} a^2(t)dW(t)],$$

as should be.