Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY 28th OCTOBER 2015, 08-13 hrs

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering.* Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 28th OCTOBER 2015 at 15.30.

The exam results will be announced at the latest on Tuesday the 12^{th} of November, 2015.

Your graded exam paper can be retained at the Student affairs office of the Department of Mathematics during a period of seven weeks after the date of the exam.

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Uppgift 1

(X, Y) is a bivariate r.v. with the joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x-xy}, & \text{for } x > 0, y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the distribution of $X(1+Y)$. (10 p)

Uppgift 2

$X \in N(0, \sigma_x^2)$ and $f(x)$ is the p.d.f. of $N(0, \sigma_c^2)$. $U \in U(0, f(0))$ and is independent of X.

a) Find the probability
$$\mathbf{P}\left(\{X \leq x\} \cap \{U \leq f(X)\}\right).$$
(4 p)

b) Find the probability
$$\mathbf{P}\left(U \leq f(X)\right).$$
(4 p)

c) Establish by the preceding that $X \mid U \leq f(X) \in N\left(0, s^2\right)$, where $\frac{1}{s^2} = \frac{1}{\sigma_x^2} + \frac{1}{\sigma_c^2}$. (2 p)

Uppgift 3

a) U has a symmetric Bernoulli distribution, i.e., its p.m.f. is

$$p_U(k) = \begin{cases} 1/2 & \text{if } k = -1, \\ 1/2 & \text{if } k = +1, \\ 0 & \text{otherwise.} \end{cases}$$

$V \in \text{Exp}(a)$ and is independent of U. Show that

$$UV \in \text{L}(a).$$

(5 p)

b) X_i, $i = 1, 2, \ldots, n$ are I.I.D., $X_i \in \text{L}(a)$. Show that

$$\frac{2\sum_{i=1}^{n} |X_i|}{a} \in \chi^2(2n).$$

(5 p)

Uppgift 4

$\{X_n\}_{n=0}^{\infty}$ is a sequence of random variables with values in the interval $[0,1]$. We set $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. We assume that $X_0 = a$, where $0 < a < 1$. Let us also assume that for $n = 0, 1, \ldots$

$$\mathbf{P}\left(X_{n+1} = \frac{X_n}{2} \mid \mathcal{F}_n\right) = 1 - X_n,$$

and

$$\mathbf{P}\left(X_{n+1} = \frac{1+X_n}{2} \mid \mathcal{F}_n\right) = X_n.$$

We know (in view of sf2940 Homework2 2015) that $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a martingale. You will need this fact, but You are not required prove this piece of knowledge anew.

a) Show now instead that

$$E\left[(X_{n+1} - X_n)^2\right] = \frac{1}{4} E\left[X_n\left(1 - X_n\right)\right].$$

(4 p)

b) It can be shown that $X_n \overset{2}{\to} X$, as $n \to +\infty$, please take this convergence for granted here. Prove that the limiting r.v. X satisfies

$$E\left[X\left(1 - X\right)\right] = 0.$$

(3 p)

c) Find the distribution of X from the equality stated in Uppgift 4 part b) above. You are expected to present your arguments in the precise detail. *Aid:* It holds for martingales that $E\left[X_n\right] = E\left[X_0\right]$ for all $n \geq 1$. (3 p)

Uppgift 5

Let $\mathbf{X} = \{X(t) \mid -\infty < t < \infty\}$ be a weakly stationary process with mean function that equals μ, and with the autocovariance function

$$\mathrm{Cov}_X(h) = \frac{1}{a} e^{-\frac{a}{2}|h|}, \quad a > 0.$$

Show that for any t

$$X\left(t + \frac{1}{n}\right) \overset{P}{\to} X(t),$$

as $n \to +\infty$. (10 p)

Uppgift 6

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process. We define the jointly Gaussian sequence (a.k.a. process in discrete time) of bivariate r.v.'s $(X_k, Y_k)^{'}$ by means of

$$X_k \stackrel{\text{def}}{=} \int_0^1 \cos(2\pi kt) dW(t), \quad k = 1, 2, \ldots,$$

and

$$Y_k \stackrel{\text{def}}{=} \int_0^1 \sin(2\pi kt) dW(t), \quad k = 1, 2, \ldots.$$

a) Find

$$\text{Cov}\left(X_k, X_l\right), \text{Cov}\left(Y_k, Y_l\right), \text{Cov}\left(X_k, Y_l\right)$$

for all values of k and l, where $k \geq 1, l \geq 1$. (2 p)

b) Explain why

$$\sum_{k=1}^{n} \frac{1}{2k-1} X_k \stackrel{2}{\to} \sum_{k=1}^{\infty} \frac{1}{2k-1} X_k \in N\left(0, \frac{\pi^2}{16}\right), \quad \text{as } n \to +\infty,$$

and why

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} Y_k \stackrel{d}{=} \sum_{k=1}^{\infty} \frac{1}{2k-1} X_k.$$

(3 p)

c) Show that

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k + \sum_{k=1}^{n} Y_k^2}} \stackrel{d}{\to} N(0, a),$$

as $n \to +\infty$. Determine explicitly the value of a. You are expected to justify your solution carefully. (5 p)

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SOLUTIONS TO THE EXAM WEDNESDAY THE 28^{th} OF OCTOBER, 2015.

Uppgift 1

Set

$$U = X, \quad V = X(1+Y).$$

Here U is clearly an auxiliary variable. Then

$$X = h_1(U,V) = U, Y = h_2(U,V) = \frac{V}{U} - 1.$$

Since $0 \leq Y = \frac{V}{U} - 1$, it holds that $V \geq U \geq 0$. We need the Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}.$$

Hence $\mid J \mid = \frac{1}{u}$ and an insertion in the formula, found in the Formelsamling,

$$f_{U,V}(u,v) = f_{X,Y}\left(h_1(u,v), h_2(u,v)\right) \mid J \mid,$$

gives

$$f_{U,V}(u,v) = e^{v}, v \geq u \geq 0,$$

i.e., the desired p.d.f is

$$f_V(v) = \int_0^v e^{-v} du = ve^{-v}, v > 0.$$

or

$$f_V(v) = \begin{cases} ve^{-v}, & \text{for } v > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

A search in the Table of Appendix B shows that $V \in \Gamma(2,1)$.
ANSWER : $\underline{X(1+Y) \in \Gamma(2,1).}$

Uppgift 2

a) The desired probability can be written as

$$\mathbf{P}\left(\{X \leq x\} \cap \{U \leq f(X)\}\right) = \int_{-\infty}^{x} \int_0^{f(t)} f_{U,X}(u,t) du dt$$

and since U and X are independent,

$$= \int_{-\infty}^{x} \int_0^{f(t)} f_U(u) f_X(t) du dt = \int_{-\infty}^{x} \int_0^{f(t)} f_U(u) du f_X(t) dt$$

$$= \int_{-\infty}^{x} \frac{f(t)}{f(0)} f_X(t)dt,$$

since $U \in (0, f(0))$, i.e., $\mathbf{P}(U \leq x) = \frac{x}{f(0)}$. We have $f(t) = \frac{1}{\sigma_c\sqrt{2\pi}} e^{-t^2/2\sigma_c^2}$ and $f(0) = \frac{1}{\sigma_c\sqrt{2\pi}}$. Hence

$$= \sigma_c\sqrt{2\pi} \int_{-\infty}^{x} \frac{1}{\sigma_c\sqrt{2\pi}} e^{-t^2/2} f_X(t)dt$$

$$= \int_{-\infty}^{x} e^{-t^2/2\sigma_c^2} \frac{1}{\sigma_x\sqrt{2\pi}} e^{-t^2/2\sigma_x^2} dt$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt.$$

It looks like this is for our purposes the most suitable way to express the desired probability.

ANSWER a): $\underline{\mathbf{P}\left(\{X \leq x\} \cap \{U \leq f(X)\}\right) = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt.}$

b) The probability here is done as above with small modification:

$$\mathbf{P}\left(U \leq f(X)\right) = \int_{-\infty}^{+\infty} \int_{0}^{f(t)} f_U(u)du f_X(t)dt$$

$$= \sigma_c\sqrt{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\sigma_c\sqrt{2\pi}} e^{-t^2/2} \frac{1}{\sigma_x\sqrt{2\pi}} e^{-t^2/2\sigma_x^2} dt$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2\sigma_c^2} e^{-t^2/2\sigma_x^2} dt$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt.$$

By the properties of the p.d.f. of $N\left(0, \frac{1}{\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}}\right)$ we have

$$\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt = \sqrt{\frac{1}{\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}}}\sqrt{2\pi}.$$

ANSWER b): $\underline{\mathbf{P}\left(U \leq f(X)\right) = \frac{1}{\sigma_x}\sqrt{\frac{1}{\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}}}.}$

c) By the definition of conditional probability

$$\mathbf{P}\left(\{X \leq x\} \mid \{U \leq f(X)\}\right) = \frac{\mathbf{P}\left(\{X \leq x\} \cap \{U \leq f(X)\}\right)}{\mathbf{P}\left(U \leq f(X)\right)}.$$

When we insert the answers from a) and b), we obtain

$$= \frac{\frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt}{\frac{1}{\sigma_x}\sqrt{\frac{1}{\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}}}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}}}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_c^2}+\frac{1}{\sigma_x^2}\right)} dt.$$

But the latest expression is $\mathbf{P}(Y \leq x)$, when $Y \in N(0, s^2)$, where $\frac{1}{s^2} = \frac{1}{\sigma_c^2} + \frac{1}{\sigma_x^2}$.

Uppgift 3

This problem hinges upon keeping in mind that if $V \in \mathrm{Exp}(a)$, then

$$\mathbf{P}(V \geq 0) = 1. \tag{1}$$

a) Take $y > 0$. Then

$$\mathbf{P}(UV \leq y) = \mathbf{P}(-V \leq y | U = -1)\, \mathbf{P}(U = -1) + \mathbf{P}(V \leq y | U = +1)\, \mathbf{P}(U = +1)$$

$$= \mathbf{P}(-V \leq y) \frac{1}{2} + \mathbf{P}(V \leq y) \frac{1}{2},$$

as U has a symmetric Bernoulli distribution and is independent of V. By simple arithmetic

$$= \mathbf{P}(V \geq -y) \frac{1}{2} + \mathbf{P}(V \leq y) \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - e^{-y/a}\right),$$

since $V \in \mathrm{Exp}(a)$ and, in view of (1), we get $\mathbf{P}(V \geq -y) = 1$ for $y > 0$. Thus the p.d.f. of UV is for $y > 0$

$$f_{UV}(y) = \frac{d}{dy}\mathbf{P}(UV \leq y) = \frac{1}{2a} e^{-y/a}$$

For $y < 0$ we get as above

$$\mathbf{P}(UV \leq y) = \mathbf{P}(V \geq -y) \frac{1}{2} + \mathbf{P}(V \leq y) \frac{1}{2},$$

and by (1) we have $\mathbf{P}(V \leq y) = 0$ for $y < 0$. Then we have, as $-y > 0$, and since $F_V(y) = 1 - e^{-y/a}$ for $y > 0$,

$$= \mathbf{P}(V \geq -y) \frac{1}{2} = (1 - F_V(-y)) \frac{1}{2} = \frac{1}{2} e^{y/a}.$$

Thus the p.d.f. of UV is for $y < 0$

$$f_{UV}(y) = \frac{d}{dy}\mathbf{P}(UV \leq y) = \frac{1}{2a} e^{y/a}$$

If we write the two results about $f_{UV}(y)$ above as a single expression, we get clearly

$$f_{UV}(y) = \frac{1}{2a} e^{-|y|/a}, \quad -\infty < y < +\infty.$$

This suffices to show that

$$UV \in \mathrm{L}(a),$$

as was claimed.

b) If X_i, $i = 1, 2, \ldots, n$ are I.I.D., $X_i \in \mathrm{L}(a)$, then the part a) of this Uppgift shows that there are I.I.D. $U_i \in$ a symmetric Bernoulli distribution, and I.I.D. $V_i \in \mathrm{Exp}(a)$, Vs and Us independent, and such that

$$X_i \stackrel{d}{=} U_i V_i, \quad i = 1, 2, \ldots, n.$$

Thus it follows

$$|X_i| \stackrel{d}{=} V_i, \quad i = 1, 2, \ldots, n.$$

We compute the characteristic function of

$$\frac{2\sum_{i=1}^{n}|X_i|}{a} = \sum_{i=1}^{n} \frac{2V_i}{a}.$$

Since V_i are I.I.D., then

$$\varphi_{\sum_{i=1}^{n} \frac{2V_i}{a}}(t) = \left(\varphi_{\frac{2V}{a}}(t)\right)^n,$$

where $V \in \mathrm{Exp}(a)$. Furthermore,

$$\left(\varphi_{\frac{2V}{a}}(t)\right)^n = \left(\varphi_V\left(\frac{2}{a}t\right)\right)^n.$$

We find from our resources in Bilaga B that

$$\varphi_V(t) = \frac{1}{1 - ait}.$$

Thus we obtain

$$\left(\varphi_V\left(\frac{2V}{a}t\right)\right)^n = \left(\frac{1}{1 - ai\frac{2}{a}t}\right)^n = \left(\frac{1}{1 - 2it}\right)^n.$$

From Bilaga B we observe now that

$$\left(\frac{1}{1 - 2it}\right)^{n/2}$$

is the characteristic function of $\chi^2(n)$. Hence it follows by uniqueness of characteristic functions that

$$\frac{2\sum_{i=1}^{n}|X_i|}{a} \in \chi^2(2n),$$

as was claimed.

Uppgift 4

a) We start by the double expectation

$$E\left[(X_{n+1} - X_n)^2\right] = E\left[E\left[(X_{n+1} - X_n)^2 \,|\mathcal{F}_n\right]\right].$$

In the inner expectation

$$E\left[(X_{n+1} - X_n)^2 \,|\mathcal{F}_n\right] = E\left[X_{n+1}^2|\mathcal{F}_n\right] - 2E\left[X_{n+1} \cdot X_n|\mathcal{F}_n\right] + E\left[X_n^2|\mathcal{F}_n\right].$$

In the middle term we first take out what is known and get

$$2E\left[X_{n+1}\cdot X_n|\mathcal{F}_n\right] = 2X_n E\left[X_{n+1}|\mathcal{F}_n\right],$$

and since $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a martingale,

$$= 2X_n^2.$$

In the last term we take again out what is known and find

$$E\left[X_n^2|\mathcal{F}_n\right] = X_n^2 E\left[1|\mathcal{F}_n\right] = X_n^2.$$

Finally, by the law of the unconscious statistician, the first term is

$$E\left[X_{n+1}^2|\mathcal{F}_n\right] = \left(\frac{X_n}{2}\right)^2(1-X_n) + \left(\frac{1+X_n}{2}\right)^2 X_n =$$

$$= \frac{X_n^2}{4} - \frac{X_n^3}{4} + \frac{X_n}{4} + \frac{2X_n^2}{4} + \frac{X_n^3}{4}$$

$$= \frac{3X_n^2}{4} + \frac{X_n}{4}.$$

When we collect the results above we have the inner expectation as

$$E\left[(X_{n+1}-X_n)^2|\mathcal{F}_n\right] = \frac{3X_n^2}{4} + \frac{X_n}{4} - 2X_n^2 + X_n^2$$

$$= -\frac{X_n^2}{4} + \frac{X_n}{4}.$$

Hence

$$E\left[(X_{n+1}-X_n)^2\right] = E\left[\frac{X_n}{4} - \frac{X_n^2}{4}\right],$$

which equals

$$= \frac{1}{4}E\left[X_n(1-X_n)\right],$$

which is as desired.

b) The fact that $X_n \overset{2}{\to} X$, as $n \to +\infty$, is an instance of the *martingale convergence theorem*, taken for granted here, and proved in many textbooks in advanced probability theory. Convergence $X_n \overset{2}{\to} X$ implies, as $n \to +\infty$,

$$E\left[X_n\right] \to E\left[X\right], E\left[X_n X_{n+1}\right] \to E\left[X^2\right]$$

and

$$E\left[X_n^2\right] \to E\left[X^2\right].$$

Hence

$$E\left[(X_{n+1}-X_n)^2\right] = E\left[X_{n+1}^2\right] - 2E\left[X_{n+1}X_n\right] + E\left[X_n^2\right]$$
$$\to E\left[X^2\right] - 2E\left[X^2\right] + E\left[X^2\right] = 0.$$

Furthermore

$$\frac{1}{4}E\left[X_n(1-X_n)\right] \to \frac{1}{4}E\left[X(1-X)\right].$$

Therefore we get

$$E\left[X(1-X)\right] = 0.$$

c) Set $Y = X(1-X)$. Since $0 \leq X_n \leq 1$ with probability one, then $0 \leq X \leq 1$. (In sf2940 Homework2 2015 it was established that if $X_n \stackrel{P}{\to} X$, as $n \to +\infty$, and $\mathbf{P}(0 \leq X_n \leq 1) = 1$, then $\mathbf{P}(0 \leq X \leq 1) = 1$. Since here $X_n \stackrel{2}{\to} X$ as $n \to +\infty$, it also holds that $X_n \stackrel{P}{\to} X$ as $n \to +\infty$.)

By $0 \leq X \leq 1$ we have that

$$Y \geq 0, \quad \text{with probability one.}$$

In addition, we have shown that

$$E[Y] = 0.$$

But, if the non negative random variable Y has zero as expectation, then the variable Y is $= 0$ with probability one. Thus

$$X(1-X) = 0 \Leftrightarrow X^2 = X \quad \text{with probability one.}$$

But the only real numbers satisfying $x^2 = x$ are $x = 1$ and $x = 0$. Hence the r.v. X has only these two values, 1 and 0. We know that a martingale has a constant mean, and therefore

$$E[X] = \lim_{n \to +\infty} E[X_n] = \lim_{n \to +\infty} E[X_0] = E[X_0] = a.$$

On the other hand, as X has only two values, 1 and 0,

$$E[X] = 1 \cdot \mathbf{P}(X=1) + 0 \cdot \mathbf{P}(X=1) = \mathbf{P}(X=1).$$

Hence we have

$$\mathbf{P}(X=1) = a.$$

Thus

ANSWER c): $\underline{X \in \text{Ber}(a).}$

Uppgift 5

If $\mathbf{X} = \{X(t) \mid -\infty < t < \infty\}$ is a weakly stationary process with mean function μ, then the autocovariance function is

$$E[(X(t)-\mu) \cdot (X(s)-\mu)] = \text{Cov}_X(t-s) = \frac{1}{a} e^{-\frac{a}{2}|t-s|}.$$

We show first that for any t

$$X\left(t + \frac{1}{n}\right) \stackrel{2}{\to} X(t),$$

as $n \to +\infty$. We consider

$$E\left[\left(X\left(t+\frac{1}{n}\right) - X(t)\right)^2\right] = E\left[\left(\left(X\left(t+\frac{1}{n}\right) - \mu\right) - (X(t)-\mu)\right)^2\right]$$

$$= E\left[\left(X\left(t+\frac{1}{n}\right)-\mu\right)^2\right] - 2E\left[\left(X\left(t+\frac{1}{n}\right)-\mu\right)\cdot(X(t)-\mu)\right] + E\left[(X(t)-\mu))^2\right].$$

By the above

$$E\left[\left((X\left(t+\frac{1}{n}\right)-\mu\right)^2\right] = \mathrm{Cov}_X(0) = \frac{1}{a},$$

$$E\left[(X\left(t\right)-\mu)^2\right] = \mathrm{Cov}_X(0) = \frac{1}{a},$$

and

$$2E\left[(X\left(t+\frac{1}{n}\right)-\mu)\cdot(X(t)-\mu)\right] = 2\mathrm{Cov}_X\left(\frac{1}{n}\right) = \frac{2}{a}e^{-\frac{a}{2}\frac{1}{n}}.$$

Hence

$$E\left[\left(X\left(t+\frac{1}{n}\right)-X(t)\right)^2\right] = \frac{1}{a} - \frac{2}{a}e^{-\frac{a}{2}\frac{1}{n}} + \frac{1}{a} = \frac{2}{a}\left(1-e^{-\frac{a}{2}\frac{1}{n}}\right).$$

As $n \to +\infty$, and $a > 0$, $\left(1-e^{-\frac{a}{2}\frac{1}{n}}\right) \to 0$, and therefore we have shown that

$$X\left(t+\frac{1}{n}\right) \stackrel{2}{\to} X(t),$$

as $n \to +\infty$. But this implies that

$$X\left(t+\frac{1}{n}\right) \stackrel{P}{\to} X(t),$$

as $n \to +\infty$, as was to be shown.

Uppgift 6

a) By the properties of the Wiener integral (c.f., Formelsamling)

$$E[X_k] = E[Y_k] = 0.$$

Furthermore, by the properties of the Wiener integral, as covered in the Formelsamling,

$$E[X_k X_l] = \int_0^1 \cos(2\pi kt)\cos(2\pi lt)dt = \begin{cases} 0 & k \neq l \\ \frac{1}{2} & k = l, \end{cases}$$

$$E[Y_k Y_l] = \int_0^1 \sin(2\pi kt)\sin(2\pi lt)dt = \begin{cases} 0 & k \neq l \\ \frac{1}{2} & k = l, \end{cases}$$

and

$$E[X_k Y_l] = \int_0^1 \cos(2\pi kt)\sin(2\pi lt)dt = 0$$

for all $k \geq 1, l \geq 1$. These results can be found with integration by parts, and are well known in mathematics (Fourier series), see L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*, chapter 13, section 13.1, p. 310. Thus

$$\mathrm{Cov}(X_k, X_l) = \begin{cases} 0 & k \neq l \\ \frac{1}{2} & k = l, \end{cases}$$

$$\mathrm{Cov}\,(Y_k, Y_l) = \begin{cases} 0 & k \neq l \\ \frac{1}{2} & k = l, \end{cases}$$

and

$$\mathrm{Cov}\,(X_k, Y_l) = 0.$$

b) Since X_k, Y_k is a jointly Gaussian sequence of r.v.'s, it holds that in view of a) that X_k are I.I.D. variables and by properties of the Wiener integral, $X_k \in N\left(0, \frac{1}{2}\right)$. Note that

$$\mathrm{Var}(X_k) = \mathrm{Cov}\,(X_k, X_k) = \frac{1}{2}$$

and

$$\mathrm{Var}(Y_l) = \mathrm{Cov}\,(Y_l, Y_l) = \frac{1}{2}.$$

The sum

$$\sum_{k=1}^{n} \frac{1}{2k-1} X_k$$

converges in mean square if and only if

$$\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} < +\infty$$

By L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*, chapter 8, section 8.6, p.194 we have

$$\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

The sum

$$\sum_{k=1}^{n} \frac{1}{2k-1} X_k$$

is in addition a Gaussian r.v.. Hence it follows by properties of mean square convergence of sums of I.I.D. normal r.v.s $X_k \in N\left(0, \frac{1}{2}\right)$ that

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} X_k \in N\left(0, \frac{\pi^2}{16}\right),$$

as $n \to +\infty$. The variance of the limit is $\mathrm{Var}[X_1] \cdot \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{1}{2} \cdot \frac{\pi^2}{8}$.

The same argument is clearly valid for $\sum_{k=1}^{\infty} \frac{1}{2k-1} Y_k$ and hence

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} Y_k \in N\left(0, \frac{\pi^2}{16}\right).$$

c) We write

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k + \sum_{k=1}^{n} Y_k^2}} = \frac{\frac{1}{\sqrt{n}\sqrt{1/2}} \sum_{k=1}^{n} X_k}{\sqrt{\frac{1}{n(1/2)} \sum_{k=1}^{n} X_k + \frac{1}{n(1/2)} \sum_{k=1}^{n} Y_k^2}}.$$

By the **Central Limit Theorem**, as $X_k \in N\left(0, \frac{1}{2}\right)$ are I.I.D. variables, we get that

$$\frac{1}{\sqrt{n}\sqrt{1/2}} \sum_{k=1}^{n} X_k \xrightarrow{d} N(0,1),$$

as $n \to +\infty$. Because $X_k \in N\left(0, \frac{1}{2}\right)$ are I.I.D. variables, the **Weak Law of Large numbers** yields

$$\frac{2}{n} \sum_{k=1}^{n} X_k \xrightarrow{P} 2 \cdot E\left[X_1\right] = 0,$$

as $n \to +\infty$. Because $Y_k \in N\left(0, \frac{1}{2}\right)$ are I.I.D. variables, the **Weak Law of Large numbers** yields

$$\frac{2}{n} \sum_{k=1}^{n} Y_k^2 \xrightarrow{P} 2 \cdot E\left[Y_1^2\right] = 2 \cdot \frac{1}{2} = 1,$$

as $n \to +\infty$. It holds also (by a theorem in the course/LN) that the sum of the two sums above converges in probability to a constant

$$\frac{2}{n} \sum_{k=1}^{n} X_k + \frac{2}{n} \sum_{k=1}^{n} Y_k^2 \xrightarrow{P} 0 + 1 = 1$$

as $n \to +\infty$. Then the **Cramér-Slutzky theorem** guarantees that

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k + \sum_{k=1}^{n} Y_k^2}} \xrightarrow{d} N(0,1).$$