Avd. Matematisk statistik

KTH Matematik

ELECTIVE HOMEWORK2 in SF2940 PROBABILITY THEORY

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Write your solutions on only one page of each sheet. You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. You will not gain points by submitting an answer without corresponding computations.
Staple your sheets of solutions together, with the homework cover sheet (handed out, downloadable) as uppermost. There can be only one student name on each submitted set of solutions.
THE DEADLINE FOR SUBMISSION: FRIDAY THE 9TH OF OCTOBER at 12.00 hours.
SUBMISSION AT LECTURES, EXERCISE CLASSES OR IN THE MAILBOX AT THE ENTRY OF THE INST.f. MATEMATIK, LINDSTEDTSVÄGEN 25.
NO ELECTRONIC SUBMISSION IS PERMITTED.

The homework will be graded and the graded solutions will be handed back NO LATER THAN THURSDAY THE 22ND OF OCTOBER (the rescheduled date of the Workshop/Räknestuga).

There are TEN (10) assignments in Homework2. The maximum number of points awarded by each assignment is conferred next to it.

The bonus points gained will be valid **in the exam 28th of October, 2015, AND in the exam 7th of January 2016.**

THE SCALE:
Bonus points in the exam −− graded points in the Homework2.

0 for 0 – 10 points,
1 for 11 – 20 points ,
2 for 21 – 30 points,
3 for 31 – 40 points,
4 for 41 – 50 points.

Bonus points from Homework1 will be added to the bonus points gained in Homework2.

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Homework2: Assignments 1.–10.

1. $\{X_n\}_{n=0}^{\infty}$ is a sequence of random variables with values in the interval $[0,1]$. We set $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. We assume that $X_0 = a$, where $0 \leq a \leq 1$. Let us also assume that for $n = 0, 1, \ldots$
$$\mathbf{P}\left(X_{n+1} = \frac{X_n}{2} \mid \mathcal{F}_n\right) = 1 - X_n,$$
and
$$\mathbf{P}\left(X_{n+1} = \frac{1+X_n}{2} \mid \mathcal{F}_n\right) = X_n.$$
Show that $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a martingale. (4 p)

2. Let $X_1, X_2, \ldots, X_n$ be I.I.D. r.v.'s. N is independent of the X_n- variables. N has the non negative integers as values. We set
$$S_N = X_1 + X_2 + \ldots + X_N.$$
Show that
$$\text{Cov}(S_N, N) = E[X] \cdot \text{Var}[N].$$
This is in fact problem 13 in section 5.8.3 in LN with a typing error corrected. (2 p)

3. X is a discrete r.v. that is uniformly distributed on the integers $\{1, 2, \ldots, n\}$, where $n > 1$. I.e., $p_X(i) = \frac{1}{n}$, $i = 1, 2, \ldots, n$. We write also $X \in U(1, \ldots, n)$. Find $E[X]$ by means of the the probability generating function (p.g.f.) of $U(1, \ldots, n)$. (2 p)

4. Y is a discrete r.v. with the positive integers as values. The probability mass function of Y is of the form
$$P(Y = k) = c \cdot \frac{1}{k!}, \quad k = 1, 2., \ldots, .$$
   (a) What is the value of c? (1 p)

   $U_1, U_2, \ldots,$ are I.I.D. r.v.s with $U_i \in U(0,1)$. $U_1, U_2, \ldots,$ are also independent of Y.

   (b) Set
$$M \overset{def}{=} \max\left(U_1, U_2, \ldots, U_Y\right).$$
   Show that for any t in the interval $[0,1]$
$$P(M \leq t) = g_Y(t),$$
   where $g_Y(t)$ is the p.g.f. of Y. Recapitulate the explicit expression for $g_Y(t)$, too. (3 p)

5. The sequence $\{X_n\}_{n=1}^{\infty}$ of random variables is such that $E[X_i] = \mu$ for all i, $\text{Cov}(X_i, X_j) = 0$, if $i \neq j$ and such that $\text{Var}(X_i) \leq c$ and for all i. Observe that the variances are uniformly bounded but not necessarily equal for all i. Show that

$$\frac{1}{n}\sum_{j=1}^{n} X_j \overset{2}{\to} \mu,$$

as $n \to \infty$. (4 p)

6. Let X have the Erlang$(n, 1)$ distribution. $Y \mid X = x \in \text{Po}(x)$.

   (a) Find the characteristic function of Y. (1 p)

   (b) Show that

$$\frac{Y - E[Y]}{\sqrt{\text{Var}[Y]}} \overset{d}{\to} N(0, 1),$$

   as $n \to +\infty$. (8 p)

7. $\Omega = (0, 1]$, $\mathcal{F}$ is the Borel σ-algebra of subsets of $(0, 1]$. $\mathbf{P}$ is the probability measure on $\mathcal{F}$ such that $\mathbf{P}([a, b]) = b - a$ for $0 < a \leq b \leq 1$. We define the sequence of r.v.'s X_n by

$$X_n(\omega) = n\omega^n, \quad n = 1, 2, \ldots, .$$

Let $n \to +\infty$.
Does the sequence $(X_n)_{n\geq 1}$ converge almost surely? Does the sequence $(X_n)_{n\geq 1}$ converge in probability? Does the sequence $(X_n)_{n\geq 1}$ converge in mean square? Does the sequence $(X_n)_{n\geq 1}$ converge in distribution? Find the limits in each case, if they exist.
You are to justify your answers carefully. (4 p)

8. $\Theta \in U(0, 2\pi)$. We set

$$X_n = \cos(n\Theta), \quad n = 1, 2, \ldots, .$$

Let $n \to +\infty$.
Does the sequence $(X_n)_{n\geq 1}$ converge almost surely? Does the sequence $(X_n)_{n\geq 1}$ converge in probability? Does the sequence $(X_n)_{n\geq 1}$ converge in mean square? Does sequence $(X_n)_{n\geq 1}$ converge in distribution? Find the limits in each case, if they exist.
You are to justify your answers carefully[1]. (5 p)

9. $\mathbf{X} = (X_1, X_2)' \in N(\mu, C)$, where

$$\mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ och } C = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

   (a) Compute

$$P(X_1 \leq 1 | X_1 - 4X_2 = 5).$$

   (3 p)

   (b) Find $E[X_1^2 X_2 \mid X_2 = 2]$. (3 p)

[1] Here the result of Assignment 10 below may turn out be useful.

10. $(X_n)_{n\geq 1}$ is a sequence of r.v.'s such that i) and ii) below are satisfied.

i) There is a real number L such that $\mathbf{P}\left(\mid X_n \mid \leq L\right) = 1$. We say that every X_n is bounded almost surely by the constant L.

ii) $X_n \stackrel{P}{\rightarrow} X$, as $n \rightarrow +\infty$.

Let us now consider the steps (a) -(d) that lead to the conclusion in (e).

(a) Show that even the limiting r.v. X is bounded almost surely by L, or,

$$\mathbf{P}\left(\mid X \mid \leq L\right) = 1.$$

Aid: Show that for any $\epsilon > 0$

$$\mathbf{P}\left(\mid X \mid \geq L + \epsilon\right) \leq \mathbf{P}(\mid X - X_n \mid \geq \epsilon).$$

and draw the desired conclusion.

(b) Justify by the preceding that $\mathbf{P}(|X - X_n|^2 \leq 4L^2) = 1$.

(c) Let I be the indicator function

$$I_{|X-X_n|\geq\epsilon} = \begin{cases} 1, \text{if } |X - X_n| \geq \epsilon \\ 0, \text{if } |X - X_n| < \epsilon \ . \end{cases}$$

Show that the inequality

$$|X - X_n|^2 \leq 4L^2 I_{|X-X_n|\geq\epsilon} + \epsilon^2$$

holds almost surely.

(d) Find now the limit of

$$E\left[\mid X - X_n \mid^2\right],$$

as $n \rightarrow +\infty$.

(e) Which theorem[2] have You hereby proved? (C.f. the summary about relations between convergences in LN pp.166−167).

(10 p)

[2]not currently stated in LN