Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY THURSDAY 7th JANUARY 2016, 14-19 hrs

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Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. L. Råde & B. Westergren: *Mathematics Handbook for Science and Engineering*. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Thursday 7th JANUARY 2016 at 21 hours.

The exam results will be announced at the latest on Wednesday the 27^{th} of January, 2016.

KINDLY WRITE THE SUM OF YOUR BONUS POINTS FROM HOMEWORK 1 AND HOMEWORK 2 ON THE COVER SHEET IN THE PLACE RESERVED FOR IT.

Your graded exam paper can be retained at the Student affairs office of the Department of Mathematics during a period of seven weeks after the date of the exam.

Lycka till!

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Uppgift 1

(X, Y) is a bivariate r.v. with the joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & \text{for } x > 0, y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a) Find the marginal distributions of X and Y. (4 p)

b) Are X and Y independent? You are expected to justify your answer. (1 p)

c) Compute $P(Y \geq a \mid X = a)$ for $a > 0$. (5 p)

Uppgift 2

X_1 and X_2 are I.I.D. and have the distribution $U(0,1)$. We define the r.v. X with $0 \leq a \leq 1$ by

$$X = \begin{cases} \max(X_1, X_2), & \text{if } X_1 < a \\ X_1, & \text{if } a \leq X_1 \leq 1. \end{cases}$$

Find the cumulative distribution function (c.d.f.) $F_X(x)$ of X. Show your calculations. You should explicitly verify that the formula you end up in presenting as the solution is in fact a c.d.f..
Aid: Consider two cases when computing $F_X(x)$: i) $0 \leq x < a$ and ii) $a \leq x \leq 1$. In the case ii) compute $F_X(x)$ by reasoning w.r.t. the events $X_1 < a$ and $X_1 \geq a$.

(10 p)

Uppgift 3

a) $X \in \text{Pa}(1, \alpha)$, $\alpha > 0$. Let $Z \stackrel{\text{def}}{=} \ln X$. Find the distribution of Z. (5 p)

b) X_i, $i = 1, 2, \ldots, n$ are I.I.D. with $\text{Pa}(1, \alpha)$. Let

$$Z_i \stackrel{\text{def}}{=} \ln X_i, i = 1, 2, \ldots, n.$$

Show that

$$2\alpha \sum_{i=1}^{n} Z_i \in \chi^2(2n).$$

(5 p)

Uppgift 4

$\mathbf{X} = \{X(t) \mid 0 \leq t < \infty\}$ is a Gaussian stochastic process with mean function $\mu_{\mathbf{X}}(t) = 0$ for all $t \geq 0$. The autocorrelation function is

$$R_X(h) = E\left[X(t)X(s)\right] = e^{-h}\cos(h), h = |t-s|.$$

Let $X_i = X(i)$, $i = 1, 2, 3$.

a) Recapitulate the expressions for $E\left[X_i \mid X_{i-1}\right]$ for $i = 2, 3$. (3 p)

b) Consider

$$Y \stackrel{\text{def}}{=} X_2 - E\left[X_2 \mid X_1\right]$$

$$Z \stackrel{\text{def}}{=} X_3 - E\left[X_3 \mid X_2\right]$$

Find the distribution of the bivariate r.v. $(Y, Z)'$. (6 p)

c) What is the distribution of $(U, V)'$, if

$$U \stackrel{\text{def}}{=} X_4 - E\left[X_4 \mid X_3\right]$$

and

$$V \stackrel{\text{def}}{=} X_5 - E\left[X_5 \mid X_4\right],$$

where $X_i = X(i)$, $i = 3, 4, 5$. Justify briefly. (1 p)

Uppgift 5

$\mathbf{X} = \{X(t) \mid 0 \leq t < \infty\}$ be a stochastic process that with probability one satisfies for every $t \geq 0$ the equation

$$X(t+h) = e^{-ah}X(t) + \int_t^{t+h} e^{-a(t+h-s)}dW(s),$$

where $h \geq 0$, $a > 0$ and $\mathbf{W} = \{W(t) \mid 0 \leq t < \infty\}$ is a Wiener process. It holds also that $E\left[X(t)\right] = 0$ and $\text{Var}\left[X(t)\right] = \frac{1}{2a}$ and that $X(t)$ is independent of $\int_t^{t+h} e^{-a(t+h-s)}dW(s)$ for any $t > 0$.
Show that for any $t > 0$

$$X\left(t + \frac{1}{n}\right) \stackrel{2}{\to} X(t),$$

as $n \to +\infty$. (10 p)

Uppgift 6

Let U_i, $i = 1, 2, \ldots$ be I.I.D. and Be(1/2) -distributed. We consider the pattern of success followed by a failure, i.e., consider the events $\{U_i = 1, U_{i+1} = 0\}$, $i = 1, 2, \ldots$. Let

$$T_n \stackrel{\text{def}}{=} \text{the number of times in } U_1, U_2, \ldots, U_n \text{ that success is followed by a failure.}$$

We introduce the indicator functions $I_{\{U_i=1, U_{i+1}=0\}}$ of the events $\{U_i = 1, U_{i+1} = 0\}$, and we set for ease of writing $I_i \stackrel{\text{def}}{=} I_{\{U_i=1, U_{i+1}=0\}}$. Then it follows that

$$T_n = \sum_{i=1}^{n-1} I_i.$$

We want to know whether $\frac{1}{n-1} T_n$ converges in probability, as n grows to infinity, i.e., we want to find, if there exists a finite number T such that

$$\frac{1}{n-1} T_n \stackrel{P}{\to} T, \quad \text{as } n \to +\infty.$$

a) Why are you NOT allowed use the Law of Large Numbers, in any of the forms you know this law from the course sf2940 in probability calculus, to find the desired limit? Explain in detail and prove your claims. (2 p)

b) Find $E[T_n]$. (1 p)

c) Show that

$$\text{Var}[T_n] = \frac{n+1}{16}.$$

(4 p)

d) Now prove that

$$\frac{1}{n-1} T_n \stackrel{P}{\to} \frac{1}{4}, \quad \text{as } n \to +\infty.$$

(3 p)

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SOLUTIONS TO THE EXAM THURSDAY THE 7^{th} OF JANUARY, 2016.

Uppgift 1

(X, Y) is a bivariate r.v. with the joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & \text{for } x > 0, y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a) The marginal distribution of X is determined by the marginal p.d.f.

$$f_X(x) = \int_0^{+\infty} xe^{-x(1+y)}dy = xe^{-x}\int_0^{+\infty} e^{-xy}dy$$

$$= xe^{-x}[\frac{e^{-xy}}{x}]_{y=0}^{+\infty} = e^{-x}.$$

The marginal distribution of Y is determined by the marginal p.d.f.

$$f_Y(y) = \int_0^{+\infty} xe^{-x(1+y)}dx = [-xe^{-x(1+y)}/(1+y)]_{x=0}^{+\infty} + \frac{1}{1+y}\int_0^{+\infty} e^{-x(1+y)}dx$$

$$= [-\frac{1}{(1+y)^2}e^{-x(1+y)}]_{x=0}^{+\infty}$$

$$= \frac{1}{(1+y)^2}.$$

ANSWER : $\underline{f_X(x) = e^{-x}, x > 0, f_Y(y) = \frac{1}{(1+y)^2}, y > 0.}$

b) X and Y are NOT independent. For example, if $x = y = 1$, then $f_{X,Y}(1,1) = e^{-2} = 0.14 \neq f_X(1) \cdot f_Y(1) = e^{-1} \cdot \frac{1}{4} = 0.092$.

c) $P(Y \geq a \mid X = a) = \int_a^{+\infty} f_{Y|X=a}(y)dy$. Here

$$f_{Y|X=a}(y) = \frac{f_{X,Y}(a,y)}{f_X(a)} = \frac{ae^{-a(1+y)}}{e^{-a}} = ae^{-ay}.$$

Hence

$$P(Y \geq a \mid X = a) = \int_a^{+\infty} ae^{-ay}dy = [-e^{-ay}]_a^{+\infty} = e^{-a^2}.$$

ANSWER : $\underline{P(Y \geq a \mid X = a) = e^{-a^2}.}$

Uppgift 2

Let $0 \leq x < a$ and consider for this x the event $\{X \leq x\}$. Then we know by construction that $\{X \leq x\}$ is not occurring via the case $X = X_1$, but as $X = \max(X_1, X_2)$. Thus

$$F_X(x) = \mathbf{P}(X \leq x) = \mathbf{P}(\max(X_1, X_2) \leq x) = \mathbf{P}(X_1 \cap X_2 \leq x)$$

and as X_1 and X_2 are independent

$$= \mathbf{P}(X_1 \leq x) \cdot \mathbf{P}(X_2 \leq x) = x \cdot x = x^2.$$

since $X_1 \in U(0,1)$ and $X_2 \in U(0,1)$.
Next, take $a \leq x \leq 1$. Then the event $\{X \leq x\}$ may occur by the event $\{X_1 \leq a\} \cap \{\max(X_1, X_2) \leq x\}$ or, by the the event $\{a \leq X_1 \leq 1\}$. These events are disjoint. Thus the additivity of probability gives

$$F_X(x) = \mathbf{P}(X \leq x) = \mathbf{P}(X_1 \leq a \cap \max(X_1, X_2) \leq x) + \mathbf{P}(a \leq X_1 \leq 1)$$

but if $x > a$, $\{X_1 \leq a\} \cap \{\max(X_1, X_2) \leq x\} = \{X_1 \leq a\} \cap \{X_2 \leq x\}$ and

$$= \mathbf{P}(X_1 \leq a \cap X_2 \leq x) + \mathbf{P}(a \leq X_1 \leq 1)$$

$$= \mathbf{P}(X_1 \leq a) \cdot \mathbf{P}(X_2 \leq x) + \mathbf{P}(a \leq X_1 \leq 1)$$

$$= a \cdot x + (x - a),$$

Thus we have found

$$F_X(x) = \begin{cases} x^2, & \text{if } 0 < x < a \\ a \cdot x + (x-a), & \text{if } a \leq x \leq 1. \end{cases}$$

The function $F_X(x)$ is non-decreasing and non-negative. Clearly, $F_X(x) = 0$ for $x \leq 0$ and $F_X(x) = 1$ for $x \geq 1$. Hence it is a cumulative distribution function.

ANSWER : $F_X(x) = \begin{cases} x^2, & \text{if } 0 < x < a \\ a \cdot x + (x-a), & \text{if } a \leq x \leq 1. \end{cases}$

Uppgift 3

a) We shall find the p.d.f. of Z. If $X \in \text{Pa}(1, \alpha)$, $\alpha > 0$, then $\mathbf{P}(X > 1) = 0$ by Appendix 2. Hence $Z = \ln X > 0$. Thus we know that the c.d.f. $F_Z(z) = \mathbf{P}(Z \leq z) = 0$ for $z \leq 0$. Take $z > 0$. Then the c.d.f. $F_Z(z)$ is

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(\ln X \leq z) = \mathbf{P}(X \leq e^z) = F_X(e^z),$$

since e^x is a monotonously increasing function of $x > 0$. Then the p.d.f is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} F_X(e^z) = f_X(e^z) \cdot e^z.$$

By Appendix 2 we have for $X \in \text{Pa}(1, \alpha)$ that the pertinent p.d.f. is

$$f_X(x) = \frac{\alpha}{x^{\alpha+1}}, x > 1.$$

Thus

$$f_Z(z) = f_X\left(e^z\right) \cdot e^z = \frac{\alpha}{e^{z(\alpha+1)}} e^z =$$
$$= \alpha e^{-z\alpha}$$

for $z > 0$. By Appendix 2 we recognize here the the exponential p.d.f. with the parameter $1/\alpha$.

ANSWER : $\underline{Z \in \text{Exp}(1/\alpha).}$

b) If X_i, $i = 1, 2, \ldots, n$ are I.I.D. with $\text{Pa}(1, \alpha)$, then by part a) $Z_i = \ln X_i, i = 1, 2, \ldots, n$ are I.I.D. $\text{Exp}(1/\alpha)$. We compute the characteristic function $\varphi_{2\alpha \sum_{i=1}^n Z_i}(t)$ of

$$2\alpha \sum_{i=1}^{n} Z_i.$$

Since Z_i are I.I.D., we have

$$\varphi_{2\alpha \sum_{i=1}^n Z_i}(t) = \varphi_{\sum_{i=1}^n 2\alpha Z_i}(t) = \left(\varphi_{2\alpha Z}\left(t\right)\right)^n,$$

where $Z \in \text{Exp}(1/\alpha)$. Furthermore,

$$\left(\varphi_{2\alpha Z}\left(t\right)\right)^n = \left(\varphi_Z\left(2\alpha t\right)\right)^n.$$

We find from our resources in Bilaga B (Appendix 2) that

$$\varphi_Z\left(t\right) = \frac{1}{1 - \frac{1}{\alpha} i t}.$$

Thus we obtain

$$\left(\varphi_{2\alpha Z}\left(t\right)\right)^n = \left(\frac{1}{1 - \frac{1}{\alpha} i 2\alpha t}\right)^n = \left(\frac{1}{1 - 2it}\right)^n.$$

This shows in view the table of characteristic functions of Bilaga B (Appendix 2) and by uniqueness of the characteristic functions that

$$2\alpha \sum_{i=1}^{n} Z_i \in \chi^2(2n).$$

Uppgift 4

a) If $\mathbf{X} = \{X(t) \mid 0 \leq t < \infty\}$ is a weakly stationary Gaussian stochastic process, since the mean function $\mu_{\mathbf{X}}(t) = 0$ for all $t \geq 0$ and as the autocorrelation function is $R_X(h) = E\left[X(t)X(s)\right] = e^{-h}\cos(h), h = |t - s|$.
Then the conditional expectation of $X_i = X(i)$ given $X_{i-1} = X(i-1)$ is by the collection of Formulas (section 9.2)

$$E\left[X_i \mid X_{i-1}\right] = E[X_i] + \rho \cdot \frac{\sigma_{X_i}}{\sigma_{X_{i-1}}}\left(X_{i-1} - E[X_{i-1}]\right).$$

Since the mean function $\mu_{\mathbf{X}}(t) = 0$ for all $t \geq 0$ we have $E[X_i] = E[X_{i-1}] = 0$. As the autocorrelation function is $R_X(h) = E\left[X(t)X(s)\right] = e^{-h}\cos(h), h = |t-s|$ we get

$$\sigma_{X_i} = \sigma_{X_{i-1}} = \sqrt{e^{-0}\cos(0)} = 1.$$

and thus $\rho = E\left[X(i)X(i-1)\right] = e^{-1}\cos(1)$. Thus

$$E\left[X_i \mid X_{i-1}\right] = e^{-1}\cos(1)X_{i-1}.$$

for $i = 2, 3$.

ANSWER a): $\underline{E\left[X_i \mid X_{i-1}\right] = e^{-1}\cos(1)X_{i-1}.}$

b) By the preceding case a) we have

$$Y = X_2 - E\left[X_2 \mid X_1\right] = X_2 - e^{-1}\cos(1)X_1$$

$$Z = X_3 - E\left[X_3 \mid X_2\right] = X_3 - e^{-1}\cos(1)X_2$$

We can write this as

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = A \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} -e^{-1}\cos(1) & 1 & 0 \\ 0 & -e^{-1}\cos(1) & 1 \end{pmatrix}.$$

Hence, as $\mathbf{X} = \{X(t) \mid 0 \leq t < \infty\}$ is a Gaussian stochastic process, the vector $(Y, Z)'$ is a linear transformation of the Gaussian vector $(X_1, X_2, X_3)'$ and thus a bivariate Gaussian r.v.. We need to find the mean vector and the covariance matrix of $(Y, Z)'$.

Clearly

$$E[Y] = E[X_2] - e^{-1}\cos(1)E[X_1] = 0, E[Z] = E[X_3] - e^{-1}\cos(1)E[X_2] = 0.$$

$$\begin{aligned} \text{Var}[Y] &= \text{Var}[X_2] + e^{-2}\cos^2(1)\text{Var}[X_1] - 2e^{-1}\cos(1)\text{Cov}(X_2, X_1) \\ &= R_X(0) + e^{-2}\cos^2(1)R_X(0) - 2e^{-1}\cos(1)R_X(1) \\ &= 1 + e^{-2}\cos^2(1) - 2e^{-2}\cos^2(1) = 1 - e^{-2}\cos^2(1). \end{aligned}$$

By weak stationarity

$$\text{Var}[Z] = \text{Var}[X_3] + e^{-2}\cos^2(1)\text{Var}[X_2] - 2e^{-1}\cos(1)\text{Cov}(X_3, X_2) = 1 - e^{-2}\cos^2(1).$$

Finally

$$\text{Cov}(Y, Z) = E\left[(X_2 - e^{-1}\cos(1)X_1)(X_3 - e^{-1}\cos(1)X_2)\right]$$

$$= E\left[X_2X_3\right] - e^{-1}\cos(1)E\left[X_2X_2\right] - e^{-1}\cos(1)E[X_1X_3] + e^{-2}\cos^2(1)E[X_1X_2]$$

$$= R_X(1) - e^{-1}\cos(1)R_X(0) - e^{-1}\cos(1)R_X(2) + e^{-2}\cos^2(1)R_X(1).$$

Note here that $R_X(2) = e^{-2}\cos(2)$.

$$= e^{-1}\cos(1) - e^{-1}\cos(1) - e^{-1}\cos(1)e^{-2}\cos(2) + e^{-2}\cos^2(1)e^{-1}\cos(1)$$

$$= e^{-3}\cos(1)(\cos^2(1) - \cos(2)).$$

Hence Y and Z are Gaussian variables with the mean vector zero and the covariance matrix

$$C = \begin{pmatrix} 1 - e^{-2}\cos^2(1) & e^{-3}\cos(1)(\cos^2(1) - \cos(2)) \\ e^{-3}\cos(1)(\cos^2(1) - \cos(2)) & 1 - e^{-2}\cos^2(1) \end{pmatrix}.$$

ANSWER b): $\underline{(Y, Z)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, C\right)}$.

c) The random vector of $(U, V)'$ is in view of a)

$$U = X_4 - E\left[X_4 \mid X_3\right] = X_4 - e^{-1}\cos(1)X_3$$

$$V = X_5 - E\left[X_5 \mid X_4\right] = X_5 - e^{-1}\cos(1)X_4$$

We can again write this as

$$\begin{pmatrix} U \\ V \end{pmatrix} = A \cdot \begin{pmatrix} X_3 \\ X_4 \\ X_5 \end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} -e^{-1}\cos(1) & 1 & 0 \\ 0 & -e^{-1}\cos(1) & 1 \end{pmatrix}.$$

Since the process $\mathbf{X} = \{X(t) \mid 0 \le t < \infty\}$ is Gaussian and weakly stationary, it is strictly stationary, which yields

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_3 \\ X_4 \\ X_5 \end{pmatrix}.$$

Thus $(U, V)' \stackrel{d}{=} (Y, Z)'$

ANSWER b): $\underline{(U, V)' \stackrel{d}{=} (Y, Z)'.}$

Uppgift 5

In view of the definition of convergence in mean square we study the expression

$$E\left[(X(t+h) - X(t))^2\right] = E\left[\left(e^{-ah}X(t) + \int_t^{t+h} e^{-a(t+h-s)}dW(s) - X(t)\right)^2\right].$$

$$= E\left[\left(\left(e^{-ah} - 1\right)X(t) + \int_t^{t+h} e^{-a(t+h-s)}dW(s)\right)^2\right].$$

We expand this to get

$$= E\left[\left(e^{-ah} - 1\right)^2 X^2(t)\right] + 2 \cdot E\left[\left(e^{-ah} - 1\right)X(t) \cdot \int_t^{t+h} e^{-a(t+h-s)}dW(s)\right]$$

$$+E\left[\left(\int_t^{t+h} e^{-a(t+h-s)}dW(s)\right)^2\right]$$

In the right hand side we obtain by the independence stated

$$E\left[\left(e^{-ah}-1\right)X(t)\cdot\int_t^{t+h} e^{-a(t+h-s)}dW(s)\right] = E\left[\left(e^{-ah}-1\right)X(t)\right]\cdot E\left[\int_t^{t+h} e^{-a(t+h-s)}dW(s)\right]$$

$$= 0\cdot 0 = 0,$$

as $E\left[\left(e^{-ah}-1\right)X(t)\right] = \left(e^{-ah}-1\right)E\left[X(t)\right] = 0$, and by a property of the Wiener integral.

The properties of the Wiener integral give the rightmost term as

$$E\left[\left(\int_t^{t+h} e^{-a(t+h-s)}dW(s)\right)^2\right] = \int_t^{t+h} e^{-2a(t+h-s)}ds = e^{-2a(t+h)}\int_t^{t+h} e^{2as}ds =$$

and

$$= e^{-2a(t+h)}[\frac{1}{2a}e^{2as}]_t^{t+h} = \frac{1}{2a}e^{-2a(t+h}\left(e^{2a(t+h)}-e^{2at}\right)$$

$$= \frac{1}{2a}\left(1-e^{-2ah}\right).$$

Finally, the first term is

$$E\left[\left(e^{-ah}-1\right)^2 X^2(t)\right] = \left(e^{-ah}-1\right)^2 E\left[X^2(t)\right] = \frac{1}{2a}\left(e^{-ah}-1\right)^2.$$

Thus we have

$$E\left[\left(X(t+h)-X(t)\right)^2\right] = \frac{1}{2a}\left(e^{-ah}-1\right)^2 + \frac{1}{2a}\left(1-e^{-2ah}\right).$$

Since $h>0$ we can take $h=\frac{1}{n}$. Then we have

$$E\left[\left(X\left(t+\frac{1}{n}\right)-X(t)\right)^2\right] = \frac{1}{2a}\left(e^{-a\frac{1}{n}}-1\right)^2 + \frac{1}{2a}\left(1-e^{-2a\frac{1}{n}}\right).$$

Since $a>0$, we find that, as $n\to\infty$

$$e^{-a\frac{1}{n}}-1\to 0,$$

and

$$1-e^{-2a\frac{1}{n}}\to 0.$$

Hence we have shown that for any $t>0$

$$X\left(t+\frac{1}{n}\right)\xrightarrow{2} X(t),$$

as $n\to+\infty$.

Uppgift 6

$$T_n = \sum_{i=1}^{n-1} I_i,$$

where $I_i = 1$ if $\{U_i = 1, U_{i+1} = 0\}$ and $I_i = 0$ otherwise.

We have

$$E[I_i] = 1 \cdot \mathbf{P}(U_i = 1, U_{i+1} = 0) = \mathbf{P}(U_i = 1)\mathbf{P}(U_{i+1} = 0) = \frac{1}{4},$$

as U_i and U_{i+1} are independent Be(1/2). Then

$$\text{Var}[I_i] = E[I_i^2] - (E[I_i])^2 = 1^2 \cdot \mathbf{P}(U_i = 1, U_{i+1} = 0) - (E[I_i])^2$$
$$= \frac{1}{4} - \frac{1}{16} = \frac{3}{16}.$$

Suppose that $|i - j| > 1$. Then I_i and I_j are independent, as they are functions of independent pairs of r.v.s, i.e. of (U_i, U_{i+1}) and (U_j, U_{j+1}), respectively. But if $j = i+1$, then I_i is a function of (U_i, U_{i+1}) and I_{i+1} is a function of (U_{i+1}, U_{i+2}), and I_i and I_{i+1} are NOT independent. We need the covariance of I_i and I_{i+1}, i.e.,

$$\text{Cov}(I_i, I_{i+1}) = E[I_i \cdot I_{i+1}] - E[I_i]E[I_{i+1}].$$

Let us note that I_i and I_{i+1} are never equal to one simultaneously, or that it always holds

$$I_i \cdot I_{i+1} = 0$$

for all i. This is seen as follows: If $I_i = 1$, then $U_{i+1} = 0$, but then $\{U_{i+1}, U_{i+2}\}$ is not a success followed by a failure, and thus $I_{i+1} = 0$. If $I_{i+1} = 1$, then $U_{i+1} = 1$, but this means that (U_i, U_{i+1}) is not a success followed by failure, so that $I_i = 0$.

Hence

$$\text{Cov}(I_i, I_{i+1}) = -E[I_i]E[I_{i+1}] = -\frac{1}{16}.$$

a) Clearly, we are NOT allowed use the Law of Large Numbers to study the limit in probability of $\frac{1}{n-1}T_n$, since what we know from the course sf2940 deal with the limit of an arithmetic mean of *independent* r.v.'s, which is not the case here.

b) By the preceding

$$E[T_n] = \sum_{i=1}^{n-1} E[I_i] = \frac{n-1}{4}.$$

c) Next, by a formula in the Collection of Formulas (section 2.6, note that this gives $n - 2$ as the upper limit of summation in the second term below), and by the argument that $|i - j| > 1$. Then I_i and I_j are independent for $|i - j| > 1$,

$$\text{Var}[T_n] = \sum_{i=1}^{n-1} \text{Var}[I_i] + 2\sum_{i=1}^{n-2} \text{Cov}(I_i, I_{i+1})$$

and from the above

$$= (n-1)\frac{3}{16} - 2(n-2)\frac{1}{16} = \frac{3(n-1) - 2(n-2)}{16}$$

$$= \frac{n+1}{16}.$$

d) Since

$$E[\frac{1}{n-1}T_n] = \frac{1}{n-1}E[T_n] = \frac{1}{4},$$

and

$$\text{Var}[\frac{1}{n-1}T_n] = \frac{1}{(n-1)^2}\text{Var}[T_n]$$

Chebyshev′s inequality gives for any $\epsilon > 0$ that

$$\mathbf{P}\left(|\frac{1}{n-1}T_n - \frac{1}{4}| > \epsilon\right) \leq \frac{1}{\epsilon^2}\text{Var}[\frac{1}{n-1}T_n] = \frac{1}{\epsilon^2(n-1)^2}\frac{n+1}{16}.$$

It holds that

$$\frac{n+1}{(n-1)^2} = \frac{n+1}{n^2-2n+1} = \frac{1+1/n}{n-2+1/n} \to 0,$$

as $n \to +\infty$. Hence we have shown by definition of convergence in probability that

$$\frac{1}{n-1}T_n \stackrel{P}{\to} \frac{1}{4}, \quad \text{as } n \to +\infty.$$