KTH Matematik

Avd. Matematisk statistik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY, WEDNESDAY OCTOBER 25, 2017, 08.00-13.00.

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Tillåtna hjälpmedel/Permitted means of assistance: Appendix 2 in A. Gut: An Intermediate Course in Probability, Formulas for probability theory SF2940, L. Råde & B. Westergren: Mathematics Handbook for Science and Engineering and pocket calculator.

All used notation must be explained and defined. Reasoning and the calculations must be so detailed that they are easy to follow. Each problem yields max 10 p. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. A preliminarily lower bound of 25 points will guarantee a passing result.

If you have received 5 bonus points from the home assignments, you may skip Problem 1(a). If you have received 10 bonus points, you may skip the whole Problem 1.

Solutions to the exam questions will be available at http://www.math.kth.se/matstat/gru/sf2940/ starting from Monday October 30, 2017.

Good luck!

Problem 1

Let $(W(t))_{t\geq 0}$ be a Wiener process. Set, for $t > 0$,

$$X_n := \sum_{j=0}^{n-1} \frac{1}{j+1} \left(W(t(j+1)) - W(tj) \right), \quad n \geq 1.$$

(a) Determine the distribution of X_n. (5 p)

(b) Show that X_n converges in distribution to some random variable X, as $n \to \infty$. Find the distribution of the limit random variable X. (5 p)

(Hint: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.)

Problem 2

Suppose that the joint characteristic function of X and Y equals

$$\varphi_{X,Y}(s,t) = E[e^{isX+itY}] = \exp\left(\alpha(e^{is}-1) + \beta(e^{it}-1) + \gamma(e^{i(s+t)}-1)\right),$$

with $\alpha > 0, \beta > 0, \gamma > 0$.

(a) Show that X and Y both have a Poisson distribution, but that $X + Y$ does not. (6 p)

(b) Are X and Y independent? (4 p)

Problem 3

Let $X_1, X_2, \ldots$ be independent, $U(0,1)$-distributed random variables, and let $N \in \mathrm{Po}(\lambda)$ be independent of $X_1, X_2, \ldots$. Set

$$Y_N := \max\{X_1, X_2, \ldots, X_N\}$$

($Y_N = 0$ when $N = 0$).

(a) Determine the distribution function and the characteristic function of Y_N. (4 p)

(b) Show that $E[Y_N] \longrightarrow 1$ as $\lambda \to \infty$. (3 p)

(c) Show that $\lambda(1 - Y_N)$ converges in distribution as $\lambda \to \infty$, and determine the limit distribution. (3 p)

Problem 4

Let X and Y be random variables such that $E[X] = 0$, $\mathrm{Var}(X) = \sigma^2 > 0$ and

$$Y \mid X = x \in N(x, a^2).$$

(a) Determine a distribution of X such that the vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ becomes normally distributed with mean $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and covariance matrix Λ. Determine μ_1, μ_2 and Λ. (6 p)

(b) Compute $E[e^X | Y = y]$. (4 p)

Problem 5

Let $X_1, X_2, \ldots$ be independent, $U(-1,1)$-distributed random variables,

(a) Show that the limit in probability of $\max_{1 \le j \le n} X_j$ is 1, and of $\min_{1 \le j \le n} X_j$ is -1, as $n \to \infty$. (6 p)

(b) Set

$$Y_n := \frac{\sum_{j=1}^n X_j}{\sqrt{n}\, \max_{1 \le j \le n} X_j}.$$

Show that Y_n converges in distribution as $n \to \infty$ ad determine the limiting distribution. (4 p)

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Suggested solutions to the exam Wednesday October 25, 2017. The problems can be solved using other methods than the suggested below.

Problem 1

(a) The Wiener process has the independent increments property. This implies that the random variables $Y_j := W(tj) - W(t(j-1)), j = 1, \dots n$, are independent. Moreover, $Y_j \in N(0, t)$. Therefore, $\frac{Y_j}{j+1} \in N(0, \frac{t}{(j+1)^2})$. Thus,

$$X_n = \sum_{j=0}^{n-1} \frac{Y_j}{j+1} \in N\left(0, t\sum_{j=0}^{n-1} \frac{1}{(j+1)^2}\right).$$

(b) We may use the characteristic function to see that

$$X_n \xrightarrow{d} N\left(0, t\sum_{j=0}^{\infty} \frac{1}{(j+1)^2}\right) = N\left(0, \frac{\pi^2 t}{6}\right).$$

Problem 2

(a) We have $X \in \text{Po}(\alpha + \gamma)$ and $Y \in \text{Po}(\beta + \gamma)$, since

$$\varphi_X(s) = \varphi_{X,Y}(s, 0) = \exp\{(\alpha + \gamma)(e^{is} - 1)\}, \quad \varphi_Y(t) = \varphi_{X,Y}(0, t) = \exp\{(\beta + \gamma)(e^{it} - 1)\}.$$

To find the distribution of $X + Y$, we compute its characteristic function:

$$\varphi_{X+Y}(s) = \varphi_{X,Y}(s, s) = \exp\{(\alpha + \beta)(e^{is} - 1) + \gamma(e^{2is} - 1)\}.$$

Clearly, it is not of the form $\exp\{\theta(e^{is} - 1)\}$, for some $\theta > 0$. Therefore, $X+Y$ is not Poisson distributed.

(b) X and Y are not independent, since $\varphi_{X,Y}(s, t) \neq \varphi_X(s)\varphi_Y(t)$.

Problem 3

First note that if $X \in U(0, 1)$ then its distribution function is $F_X(x) = x$, $x \in (0, 1)$. Moreover, the probability generation function of $N \in \text{Po}(\lambda)$ is $g_N(t) = E[t^N] = e^{\lambda(t-1)}$. Since the $X_1, X_2, \dots$ are i.i.d. $U(0, 1)$ and are independent of N, by conditioning on N, we have

$$F_{Y_N}(x) = P(\max\{X_1, X_2, \dots, X_N\} \leq x) = E[(F_{X_1}(x))^N] = g_N(F_{X_1}(x)) = e^{\lambda(x-1)}.$$

More precisely,

$$F_{Y_N}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{\lambda(x-1)} & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Thus, F_{Y_N} has two parts: a degenerate part, 0, which corresponds to the case where $Y_N = 0$ (which occurs when $N = 0$) and a smooth part, $e^{\lambda(x-1)}$ when $x > 0$.
To obtain the characteristic function, we first derive the differential of the smooth part of $F_{Y_N}(x)$. This is

$$\frac{d(e^{\lambda(x-1)})}{dx} = \begin{cases} \lambda e^{\lambda(x-1)} & \text{if } 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

We have

$$\varphi_{Y_N}(t) = E[e^{itY_N}] = \int e^{itx} dF_{Y_N}(x) = P(N=0) + \int_0^1 e^{itx} \lambda e^{\lambda(x-1)} dx = e^{-\lambda} + \frac{\lambda}{it+\lambda}\left(e^{it} - e^{-\lambda}\right).$$

(b) $E[Y_N] = \frac{1}{i}\varphi'_{Y_N}(0) = 1 - \frac{1}{\lambda}\left(1 - e^{-\lambda}\right)$. Therefore, $E[Y_N] \longrightarrow 1$ as $\lambda \to \infty$.

(c) We have

$$\varphi_{\lambda(1-Y_N)}(t) = E[e^{it\lambda(1-Y_N)}] = e^{it\lambda} E[e^{-it\lambda Y_N}] = e^{it\lambda}\varphi_{Y_N}(-t\lambda) = \frac{1}{1-it}\left(1 - e^{-\lambda(1-it)}\right).$$

Since $|e^{-\lambda(1-it)}| = e^{-\lambda} \to 0$, as $\lambda \to \infty$, it holds that $e^{-\lambda(1-it)} \to 0$, as $\lambda \to \infty$. Therefore,

$$\varphi_{\lambda(1-Y_N)}(t) \longrightarrow \frac{1}{1-it} = \varphi_{\mathrm{Exp}(1)}(t).$$

Thus, $\lambda(1 - Y_N) \xrightarrow{d} \mathrm{Exp}(1)$.

Problem 4

(a) We compute the characteristic function of $(X,Y)'$ in two ways: First, we use conditioning to obtain

$$\begin{aligned} \varphi_{X,Y}(s,t) &:= E[e^{isX+itY}] = E[E[e^{isX+itY}|X]] = E[e^{isX}E[e^{itY}|X]] = (Y \mid X = x \in N(x,a^2)) \\ &= E[e^{isX}e^{itX-\frac{a^2t}{2}}] = e^{-\frac{a^2t}{2}}E[e^{i(s+t)X}]. \end{aligned}$$

That is

$$E[e^{i(s+t)X}] = e^{\frac{a^2t}{2}}\varphi_{X,Y}(s,t).$$

Next, we use the assumption that $(X,Y)' \in N(\mu, \Lambda)$, where $\mu = (\mu_1, \mu_2)'$, with $\mu_1 = E[X] = 0$, and $\Lambda = \begin{pmatrix} \sigma^2 & c \\ c & b^2 \end{pmatrix}$, where $\sigma^2 = \mathrm{Var}(X)$ and $c := \mathrm{Cov}(X,Y)$, $b^2 = \mathrm{Var}(Y)$ to be determined together with $\mu_2 = E[Y]$.
We have

$$\begin{aligned} \varphi_{X,Y}(s,t) &= \exp\left\{i(s\mu_1 + t\mu_2) - \tfrac{1}{2}(\sigma^2s^2 + b^2t^2 + 2cst)\right\} = (\mu_1 = 0) \\ &= \exp\left\{it\mu_2 - \tfrac{1}{2}(\sigma^2s^2 + b^2t^2 + 2cst)\right\}. \end{aligned}$$

Therefore,

$$E[e^{i(s+t)X}] = \exp\left\{it\mu_2 - \frac{1}{2}(\sigma^2 s^2 + (b^2 - a^2)t^2 + 2cst)\right\}$$

The assumption that $(X, Y)' \in N(\mu, \Lambda)$ implies that $X \in N(E[X], \text{Var}(X)) = N(0, \sigma^2)$. Therefore,

$$E[e^{i(s+t)X}] = \exp\left\{-\frac{1}{2}\sigma^2(s+t)^2\right\}.$$

Thus, we should have

$$e^{-\frac{1}{2}\sigma^2(s+t)^2} = \exp\left\{it\mu_2 - \frac{1}{2}(\sigma^2 s^2 + (b^2 - a^2)t^2 + 2cst)\right\}, \quad \forall s, t.$$

Identifying the coefficients, we obtain that $\mu_2 = 0,\ c = \sigma^2$ and $b^2 = a^2 + \sigma^2$. Hence, $(X, Y)' \in N(\mu, \Lambda)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & a^2 + \sigma^2 \end{pmatrix}.$$

(b) To compute $E[e^X | Y = y]$, we note that $X \,|\, Y = y \in N(\mu_{1|2}, \sigma^2_{1|2})$, where

$$\mu_{1|2} = 0 + \frac{\sigma^2}{a^2 + \sigma^2}(y - 0) = \frac{\sigma^2}{a^2 + \sigma^2} y, \quad \sigma^2_{1|2} = \sigma^2 - \frac{\sigma^4}{a^2 + \sigma^2}.$$

Therefore,

$$E[e^X | Y = y] = e^{\mu_{1|2} + \frac{1}{2}\sigma^2_{1|2}} = \exp\left(\frac{\sigma^2}{a^2 + \sigma^2} y + \frac{1}{2}(\sigma^2 - \frac{\sigma^4}{a^2 + \sigma^2})\right).$$

Problem 5

Before, answering (a) and (b), we note that if $X \in U(-1, 1)$ then

$$F_X(x) = P(X \le x) = \frac{x+1}{2}, \ -1 < x < 1, \quad E[X] = 0, \quad \text{Var}(X) = 1/3.$$

(a) Pick $\epsilon > 0$ and set $X_{(1)} := \min_{1 \le j \le n} X_j$ and $X_{(n)} := \max_{1 \le j \le n}$. we have

$$P(|X_{(n)} - 1| > \epsilon) = P(X_{(n)} > 1 + \epsilon) + P(X_{(n)} < 1 - \epsilon).$$

But, since all $X_j \in (-1, 1)$, then $X_{(n)} \in (-1, 1)$. This implies that $P(X_{(n)} > 1 + \epsilon) = 0$. Thus,

$$P(|X_{(n)} - 1| > \epsilon) = P(X_{(n)} < 1 - \epsilon) = (X_j's \text{ are i.i.d}) = (F_{X_1}(1 - \epsilon))^n = 2^{-n}(1 + 1 - \epsilon)^n = \left(\frac{2 - \epsilon}{2}\right)^n.$$

This, in turn suggest that $0 < \epsilon < 2$ (otherwise we get negative probabilities), which implies that $P(|X_{(n)} - 1| > \epsilon) \longrightarrow 0$ as $n \to \infty$, i.e. $X_{(n)} \xrightarrow{P} 1$.

We have

$$P(|X_{(1)} + 1| > \epsilon) = P(X_{(1)} > \epsilon - 1) + P(X_{(1)} < -(\epsilon + 1)).$$

But, $P(X_{(1)} < -(\epsilon + 1)) = 0$, as all $X_j \in (-1, 1)$. Moreover,

$$P(X_{(1)} > \epsilon - 1) = (1 - F_{X_1}(\epsilon - 1))^n = \left(1 - \frac{1 + \epsilon - 1}{2}\right)^n = \left(\frac{2 - \epsilon}{2}\right)^n.$$

Therefore,

$$P(|X_{(1)} + 1| > \epsilon) = \left(\frac{2 - \epsilon}{2}\right)^n \longrightarrow 0, \quad n \to \infty,$$

i.e. $X_{(1)} \xrightarrow{P} -1$.

(b) By the Central Limit Theorem, it holds that $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \xrightarrow{d} N(0, \frac{1}{3})$. Moreover, by (a) $X_{(n)} \xrightarrow{P} 1$. Use Cramèr-Slutsky's theorem to obtain

$$Y_n \xrightarrow{d} N\left(0, \frac{1}{3}\right).$$