KTH Matematik

Avd. Matematisk statistik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY, Tuesday December 19, 2017, 08.00-13.00.

Examinator : Boualem Djehiche, tel. 08-7907875, email: boualem@kth.se

Tillåtna hjälpmedel/Permitted means of assistance: Appendix 2 in A. Gut: An Intermediate Course in Probability, Formulas for probability theory SF2940, L. Råde & B. Westergren: Mathematics Handbook for Science and Engineering and pocket calculator.

All used notation must be explained and defined. Reasoning and the calculations must be so detailed that they are easy to follow. Each problem yields max 10 p. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. 25 points will guarantee a passing result.

If you have received 5 bonus points from the home assignments, you may skip Problem 1(a). If you have received 10 bonus points, you may skip the whole Problem 1.

The bonus points system does not apply for those who already passed the exam and want to improve their grades.

Solutions to the exam questions will be available at the course's homepage.

Good luck!

## Problem 1

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables such that $Y_i \in \text{Po}(1)$ i.e. Poisson distributed with parameter 1. Set

$$S_n := Y_1 + Y_2 + \cdots + Y_n.$$

(a) Determine the limit distribution of

$$\frac{S_n - n}{\sqrt{n}},$$

as $n \to \infty$. All the steps should be carefully justified. (5 p)

(b) Find

$$\lim_{n\to\infty} P(S_n \leq n).$$

(5 p)

## Problem 2

Suppose that the characteristic function of $(X, Y)$ is

$$\varphi_{X,Y}(\alpha, \beta) = \exp\{2i\alpha + 3i\beta - \alpha^2 - \lambda\alpha\beta - 2\beta^2\}.$$

Determine $\lambda$ so that $X+2Y$ and $2X-Y$ become independent. (10 p)

**Problem 3**

Let $X \in N(0,\Lambda)$, where

$$\Lambda = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

Set $Y_1 = X_1+X_3,\ Y_2 = 2X_1-X_2$, and $Y_3 = 2X_3-X_2$. Compute the conditional expectations $E[Y_3|Y_1=3]$ and $E[Y_3|Y_2=-1]$. (10 p)

**Problem 4**

The random variables $X_1, X_2, \ldots,$ are independent and identically distributed (i.i.d.) with mean $\mu$ and variance $\sigma^2$. Set $\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$.

(a) Verify that

$$\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \sum_{i=1}^n (X_i-\mu)^2 - n(\overline{X}_n-\mu)^2.$$

(1 p)

(b) Let

$$S_n^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2.$$

Show that

$$S_n \xrightarrow{P} \sigma, \quad \text{as } n \to \infty.$$

(3 p)

(c) Find the limit distribution of

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{S_n},$$

as $n \to \infty$. All the steps should be carefully justified. (6 p)

**Problem 5**

For $s>0$, let $Y_s$ be $N(0,s)$-distributed. Let $T \in Exp(\frac{1}{\lambda})$ be independent of all $Y_s,\ s>0$.

(a) Compute the characteristic function of $Y_T$ and determine its distribution. (6 p)

(b) Determine the limit distribution of $Y_T$ and $\sqrt{\lambda}Y_T$, as $\lambda \to \infty$. (4 p)

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Suggested solutions to the exam Tuesday December 19, 2017, 08.00-13.00.

## Problem 1

(a) For $Y \in \text{Po}(1)$, we have $E[Y] = \text{Var}(Y) = 1$. Since $Y_1, Y_2, \ldots$ are i.i.d., we may apply the Central Limit Theorem to the sum $S_n := \sum_{i=1}^n X_i$ to get that

$$\frac{S_n - n}{\sqrt{n}} \xrightarrow{d} N(0,1),$$

or, for every $x \in \mathbb{R}$

$$P\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) \longrightarrow F_{N(0,1)}(x) := \Phi(x).$$

(b) We have

$$\lim_{n\to\infty} P(S_n \leq n) = P\left(\frac{S_n - n}{\sqrt{n}} \leq 0\right) = F_{N(0,1)}(0) = \Phi(0) = \frac{1}{2}.$$

## Problem 2

Set $Z := X + 2Y$ and $W := 2X - Y$. We have to choose $\lambda$ such that

$$\varphi_{Z,W}(t,u) = E[e^{itZ+iuW}] = f(t)g(u)$$

where $f$ and $g$ are characteristic functions (necessarily of $Z$ and $W$ respectively). We have

$$\varphi_{Z,W}(t,u) = E[e^{itZ+iuW}] = E[e^{it(X+2Y)+iu(2X-Y)}] = E[e^{i(t+2u)X+i(2t-u)Y}] = \varphi_{(X,Y)}(t+2u, 2t-u).$$

This yields

$$\begin{aligned} &= \varphi_{(X,Y)}(t+2u, 2t-u) = \exp\{2i(t+2u) + 3i(2t-u) - (t+2u)^2 - \lambda(t+2u)(2t-u) - 2(2t-u)^2\} \\ &= \exp\{8it + iu - (9+2\lambda)t^2 - (6-2\lambda)u^2 - (3\lambda - 4)tu\}. \end{aligned}$$

We see that if $\lambda = 4/3$ we obtain

$$\varphi_{Z,W}(t,u) = \exp\{8it - \frac{35}{3}t^2 + iu - \frac{10}{3}u^2\} = \exp\{8it - \frac{35}{3}t^2\}\exp\{iu - \frac{10}{3}u^2\} := f(t)g(u),$$

where

$$f(t) = \exp\{8it - \frac{35}{3}t^2\} = \varphi_{N(8,\frac{70}{3})}(t), \quad g(u) = \exp\{iu - \frac{10}{3}u^2\} = \varphi_{N(1,\frac{20}{3})}(u),$$

that is $Z \in N(8, \frac{70}{3})$ and $W \in N(1, \frac{20}{3})$ and are independent.

**Problem 3**

We have $\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma\right)$ where

$$\Sigma = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 2 & -2 \\ 4 & -2 & 22 \end{pmatrix}.$$

Thus,

$$E[Y_3|Y_1 = 3] = 0 + \frac{4}{3}(3-0) = 4, \quad E[Y_3|Y_2 = -1] = 0 + \frac{-2}{2}(-1-0) = 1.$$

**Problem 4**

(a) We have

$$\begin{aligned} \textstyle\sum_{i=1}^n (X_i - \overline{X}_n)^2 &= \textstyle\sum_{i=1}^n (X_i - \overline{X}_n - \mu + \mu)^2 \\ &= \textstyle\sum_{i=1}^n (X_i - \mu)^2 + (\overline{X}_n - \mu)^2 - 2(X_i - \mu)(\overline{X}_n - \mu) \\ &= \textstyle\sum_{i=1}^n (X_i - \mu)^2 + n(\overline{X}_n - \mu)^2 - 2(\overline{X}_n - \mu)\sum_{i=1}^n (X_i - \mu) \\ &= \textstyle\sum_{i=1}^n (X_i - \mu)^2 + n(\overline{X}_n - \mu)^2 - 2n(\overline{X}_n - \mu) \\ &= \textstyle\sum_{i=1}^n (X_i - \mu)^2 - n(\overline{X}_n - \mu)^2, \end{aligned}$$

since $\sum_{i=1}^n (X_i - \mu) = \sum_{i=1}^n X_i - n\mu = n(\overline{X}_n - \mu)$.

(b) From (a) we obtain

$$S_n^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \mu)^2 - \frac{n}{n-1}(\overline{X}_n - \mu)^2.$$

Now, since $X_1, X_2, \ldots$ are i.i.d. with mean $\mu$ and variance $\sigma^2$, the random varibales $Y_1, Y_2, \ldots$ defined by $Y_i = (X_i - \mu),\ i = 1, 2, \ldots$ are also i.i.d. with mean

$$E[Y_i] = E[(X_i - \mu)^2] = \text{Var}(X_i) = \sigma^2, \qquad i = 1, 2, \ldots$$

Using the law of large numbers, we obtain

$$\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 \xrightarrow{P} \sigma^2, \quad \text{as } n \to \infty.$$

Therefore, using the fact that $\frac{n}{n-1} \to 1$ as $n \to +\infty$, we finally obtain

$$\frac{1}{n-1}\sum_{i=1}^n (X_i - \mu)^2 = \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 \xrightarrow{P} E[(X-\mu)^2] = \text{Var}(X) = \sigma^2.$$

Again by the law of large numbers, as $n \to +\infty$,

$$\overline{X}_n - \mu = \frac{1}{n}\sum_{i=1}^n X_i - \mu \xrightarrow{P} 0.$$

Thus, as $n \to +\infty$,

$$\frac{n}{n-1}(\overline{X}_n - \mu)^2 \xrightarrow{P} 0.$$

Summing up, $S_n^2 \xrightarrow{P} \sigma^2$.

(c) We have

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{S_n} = \frac{\sigma}{S_n}\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma}.$$

But, in view of the Central Limit Theorem,

$$\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} = \sqrt{n}\frac{1}{n\sigma}\left(\sum_{i=1}^{n} X_i - n\mu)\right) = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

We may apply Cramér-Slutszky's theorem to obtain that

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{S_n} \xrightarrow{d} N(0,1),$$

as $n \to +\infty$.

## Problem 5

(a) We have

$$\varphi_{Y_T}(t) = E[e^{itY_T}] = \int_0^\infty E[e^{itY_T}|T = s]f_T(s)ds = \int_0^\infty E[e^{itY_s}|T = s]f_T(s)ds.$$

Since $T \in Exp(\frac{1}{\lambda})$ and $Y_s \in N(0,s)$ are independent, we obtain

$$\varphi_{Y_T}(t) = E[e^{itY_T}] = \int_0^\infty E[e^{itY_s}]f_T(s)ds = \int_0^\infty e^{-\frac{t^2}{2}s}\lambda e^{-\lambda s}ds = \frac{\lambda}{\lambda + \frac{t^2}{2}} = \frac{1}{1 + \frac{t^2}{2\lambda}}.$$

which is the characteristic function of the Laplace distribution $\mathrm{La}(\frac{1}{\sqrt{2\lambda}})$.

(b) We have, as $\lambda \to \infty$,

$$\varphi_{Y_T}(t) = \frac{1}{1 + \frac{t^2}{2\lambda}} \to 1,$$

where 1 is the characteristic function of the random variable $X = 0$. Thus, $Y_T \xrightarrow{d} 0$.

$$\varphi_{\sqrt{\lambda}Y_T}(t) = \varphi_{\sqrt{\lambda}Y_T}(t\sqrt{\lambda}) = \frac{1}{1 + \frac{(t\sqrt{\lambda})^2}{2\lambda}} = \frac{1}{1 + \frac{t^2}{2}} = \varphi_{\mathrm{La}(\frac{1}{\sqrt{2}})}.$$

This means that $\sqrt{\lambda}Y_T \in \mathrm{La}(\frac{1}{\sqrt{2}})$ for all $\lambda > 0$.