KTH Matematik

Avd. Matematisk statistik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY, WEDNESDAY OCTOBER 24, 2018, 08.00-13.00.

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Tillåtna hjälpmedel/Permitted means of assistance: Appendix 2 in A. Gut: An Intermediate Course in Probability, Formulas for probability theory SF2940, L. Råde & B. Westergren: Mathematics Handbook for Science and Engineering and pocket calculator.

All used notation must be explained and defined. Reasoning and the calculations must be so detailed that they are easy to follow. Each problem yields max 10 p. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. A preliminarily lower bound of 25 points will guarantee a passing result.
Solutions to the exam questions will be available at http://www.math.kth.se/matstat/gru/sf2940/ starting from Monday October 29, 2018.

Good luck!

Problem 1

Let $X = (X_1, X_2)' \in N(\mu, \Lambda)$, where

$$\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Compute $P(X_1 \geq 2 | X_2 + 3X_1 = 1)$. (3 p)

(b) Find $E[X_1^2 X_2^4 | X_2 = 2]$. (2 p)

(c) Assume that the characteristic function of $(X, Y, Z)'$ is

$$\varphi(t, s, u) = \exp\big(2is - s^2 - 2t^2 - 4u^2 - 2st + 2su\big).$$

Find the distribution of the vector (X, Y, Z). (5 p)

Problem 2

The random variables $X \in U(0, 1)$ and $Y \in U(0, 1)$ are independent. Set $U = X + Y$ and $V = X - Y$.

(a) Find the characteristic functions $\varphi_U(t)$ and $\varphi_V(t)$ of U and V, respectively. (5 p)

(b) Show that U and V are not independent random variables. (5 p)

Problem 3

Let $X_1, X_2, \ldots$ be independent, $U(0,1)$-distributed random variables.

(a) Determine the limit in probability of

$$\frac{\sum_{j=1}^n X_j^2}{\sum_{j=1}^n X_j} \quad \text{as } n \to +\infty.$$

(5 p)

(b) Determine the limit in distribution of

$$\sqrt{n}\frac{\sum_{j=1}^n (X_j^2 - 1/3)}{\sum_{j=1}^n X_j} \quad \text{as } n \to +\infty.$$

(5 p)

Problem 4

The vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ is normally distributed with mean $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and covariance matrix $\Lambda = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where ρ is a non-zero constant.

(a) Find the constant a such that the random variable $Z := X + aY$ is independent of Y. (4 p)

(b) Determine the random variable $E[e^X | e^Y]$. (6 p)

Problem 5

Let $X_1, X_2, \ldots$ be independent, $U(0,1)$-distributed random variables, and let $N \in \mathrm{Po}(\lambda)$ be independent of $X_1, X_2, \ldots$. Set

$$Y_N := \min\{X_1, X_2, \ldots, X_N\}, \quad Y_N = 0 \text{ when } N = 0.$$

(a) Determine the distribution function and the characteristic function of Y_N. (4 p)

(b) Show that $E[Y_N] \longrightarrow 0$ as $\lambda \to \infty$. (3 p)

(c) Show that λY_N converges in distribution as $\lambda \to \infty$, and determine the limit distribution. (3 p)

Note: In a previous version of the exam, the statement (c) claimed that $\lambda(1+Y_N)$ converges in distribution as $\lambda \to \infty$. This limit does not exist.

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Suggested solutions to the exam Wednesday October 24, 2018. The problems can be solved using other methods than the suggested below.

Problem 1

The vector $(X_1, X_2 + 3X_1)$ is normally distributed with mean and covariance matrix

$$\nu = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 3 & 10 \\ 10 & 35 \end{pmatrix}.$$

(a) The conditional law of X_1 given $X_2 + 3X_1$ is normal with parameters

$$\mu_{1|2} = 1 + \frac{10}{35}(1-4) = \frac{1}{7}, \quad \sigma^2_{1|2} = 3 - \frac{100}{35} = \frac{1}{7}.$$

Therefore,

$$P(X_1 \geq 2 | X_2 + 3X_1 = 1) = 1 - \Phi\left(\frac{2-\mu_{1|2}}{\sigma_{1|2}}\right) = 1 - \Phi\left(\frac{13}{\sqrt{7}}\right),$$

where Φ is the cumulative distribution function of the standard normal $N(0,1)$.

(b) The conditional law of X_1 given X_2 is normal with parameters

$$\mu_{1|2} = 1 + \frac{1}{2}(2-1) = \frac{3}{2}, \quad \sigma^2_{1|2} = 3 - \frac{1}{2} = \frac{5}{2}.$$

Therefore,

$$E[X_1^2 X_2^4 | X_2 = 2] = 2^4 E[X_1^2 | X_2 = 2] = 16\left(\left(\frac{3}{2}\right)^2 + \frac{5}{2}\right) = 76.$$

(c) Write $\varphi(t,s,u) := E\left[e^{itX+isY+iuZ}\right]$ and identify the coefficients to see that (X,Y,Z) is normally distributed with parameters

$$\mu = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & -2 \\ 0 & -2 & 8 \end{pmatrix}.$$

Problem 2

Recall that the characteristic function of $X, Y \in U(0,1)$ is $\varphi_X(t) = \varphi_Y(t) = \frac{e^{it}-1}{it}$. We have

(a)

$$\varphi_U(t) = E[e^{it(X+Y)}] = (X, Y \text{are independent}) = \varphi_X(t)\varphi_Y(t) = \left(\frac{e^{it}-1}{it}\right)^2$$

and

$$\varphi_V(t) = E[e^{it(X-Y)}] = (X, Y \text{are independent}) = \varphi_X(t)\varphi_Y(-t) = \frac{e^{it}-1}{it}\frac{e^{-it}-1}{-it} = \frac{2-e^{it}-e^{-it}}{t^2}.$$

(b) U and V are not independent. Indeed,

$$\varphi_{(U,V)}(t,s) = E[e^{itU+isV}] = E[e^{i(t+s)X+i(t-s)Y}] = \varphi_X(t+s)\varphi_Y(t-s) = \frac{e^{i(t+s)}-1}{i(t+s)}\frac{e^{i(t-s)}-1}{i(t-s)}$$

is not equal to $\varphi_U(t)\varphi_V(s)$: for $t=2, s=1$, $\varphi_{(U,V)}(2,1) = 0 \neq \varphi_U(2)\varphi_V(1)$.

Problem 3

Since the X_j's have the same distribution as $X \in U(0,1)$ it holds that they have the same mean $\mu = E[X] = \frac{1}{2}$ and the same second moment $E[X^2] = \frac{1}{3}$.

(a) By the law of large numbers we have the following convergence:

$$\frac{1}{n}\sum_{j=1}^n X_j^2 \xrightarrow{P} E[X^2] = \frac{1}{3}, \qquad \frac{1}{n}\sum_{j=1}^n X_j \xrightarrow{P} E[X] = \frac{1}{2}, \quad \text{as } n \to +\infty.$$

Since the limits are constants convergence in probability is equivalent to convergence in distribution. We may use Cramèr-Slutsky's Theorem to see that

$$\frac{\sum_{j=1}^n X_j^2}{\sum_{j=1}^n X_j} \xrightarrow{d} \frac{2}{3} \quad \text{as } n \to +\infty.$$

Since the limit $2/3$ is a constant, the convergence is also in probability.

(b) We first apply the central limit theorem to $\sum_{j=1}^n X_j^2$. Indeed, we first compute the variance of X^2:

$$\hat{\sigma}^2 := \text{Var}(X^2) = E[X^4] - (E[X^2])^2 = \int_0^1 x^4 dx - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

We have

$$\frac{\sum_{j=1}^n \left(X_j^2 - \frac{1}{3}\right)}{\hat{\sigma}\sqrt{n}} \xrightarrow{d} N(0,1).$$

Again by Cramèr-Slutsky's Theorem we obtain that

$$\sqrt{n}\frac{\sum_{j=1}^n (X_j^2 - 1/3)}{\sum_{j=1}^n X_j} = \frac{1}{\sqrt{n}}\frac{\sum_{j=1}^n (X_j^2 - 1/3)}{\frac{1}{n}\sum_{j=1}^n X_j} \xrightarrow{d} 2\hat{\sigma}N(0,1) = N(0, \frac{16}{45}), \quad \text{as } n \to +\infty.$$

Problem 4

(a) First we note that

$$\rho = \mathrm{Cov}(X,Y) = \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}\mathrm{corr}(X,Y) = \mathrm{corr}(X,Y).$$

Therefore, $|\rho| \leq 1$.

(Y,Z) is normally distributed with parameters

$$\nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & a+\rho \\ a+\rho & 1+a(a+2\rho) \end{pmatrix},$$

provided that $1+a(a+2\rho) \geq 0$. Hence, Z and Y are independent if and only if $\mathrm{Cov}(Y,Z) = 0$ i.e. $a = -\rho$. The variance of Z becomes $\mathrm{Var}(Z) = (1-\rho^2)$ which is non-negative since we have shown that $|\rho| \leq 1$.

(b) We use the independence between Y and $Z = X - \rho Y$ to obtain

$$\begin{aligned} E[e^X|e^Y] &= E[e^{Z+\rho Y}|e^Y] = e^{\rho Y}E[e^Z|e^Y] \\ &= (Z \text{ and } Y \text{ independent}) = e^{\rho Y}E[e^Z] = e^{\frac{(1-\rho^2)}{2}}e^{\rho Y}. \end{aligned}$$

Problem 5

First note that if $X \in U(0,1)$ then its distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \in (0,1), \\ 1 & \text{if } \ x > 1. \end{cases}$$

Moreover, the probability generation function of $N \in \mathrm{Po}(\lambda)$ is $g_N(t) = E[t^N] = e^{\lambda(t-1)}$. Since the $X_1, X_2, \ldots$ are i.i.d. $U(0,1)$ and are independent of N, by conditioning on N, we have

$$\begin{aligned} \bar{F}_{Y_N}(x) &= P(Y_N > x) = \textstyle\sum_{n=0}^{+\infty} P(Y_N > x, N = n) \\ &= P(Y_N > x, N = 0) + \textstyle\sum_{n=1}^{+\infty} P(Y_N > x, N = n) \\ &= P(Y_0 > x, N = 0) + \textstyle\sum_{n=1}^{+\infty} P(\max\{X_1, X_2, \ldots, X_N\} > x) \\ &= P(Y_0 > x, N = 0) + \textstyle\sum_{n=1}^{+\infty}(1 - F_X(x))^n P(N = n) \\ &= P(Y_0 > x, N = 0) + e^{-\lambda}\textstyle\sum_{n=1}^{+\infty}(1 - F_X(x))^n \frac{\lambda^n}{n!} \\ &= P(Y_0 > x, N = 0) + e^{-\lambda F_X(x)} - e^{-\lambda}. \end{aligned}$$

Now, since $Y_N = 0$ if $N = 0$, when $x < 0$, $P(Y_0 > x, N = 0) = P(N = 0) = e^{-\lambda}$ and when $x \geq 0$, $P(Y_0 > x, N = 0)) = 0$.
Summing up,

$$F_{Y_N}(x) = 1 - \bar{F}_{Y_N}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 + e^{-\lambda} - e^{-\lambda x} & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Therefore, F_{Y_N} is discontinuous at 0: $F_{Y_N}(0) = e^{-\lambda} \neq F_{Y_N}(0^-) = 0$.

The characteristic function of Y_N is computed as follows.

$$\begin{aligned}\varphi_{Y_N}(t) &= E[e^{itY_N}] = \textstyle\sum_{n=0}^{+\infty} E[e^{itY_N} I_{\{N=n\}}] \\ &= E[e^{itY_N} I_{\{N=0\}}] + \textstyle\sum_{n=1}^{+\infty} E[e^{itY_N} I_{\{N=n\}}] \\ &= E[e^{itY_0} I_{\{N=0\}}] + \textstyle\sum_{n=1}^{+\infty} E[e^{itY_n} I_{\{N=n\}}] \\ &= P(N=0) + \textstyle\sum_{n=1}^{+\infty} E[e^{itY_n}] P(N=n), \quad (Y_n \text{ and } N \text{ are independent}).\end{aligned}$$

But, for $n \geq 1$,

$$F_{Y_n}(x) = 1 - P(Y_n > x) = \begin{cases} 0 & \text{if } x < 0, \\ 1-(1-x)^n & \text{if } x \in (0,1), \\ 1 & \text{if } \ x > 1. \end{cases}$$

Therefore, the probability density of Y_n is $f_{Y_n}(x) = n(1-x)^{n-1}, \quad 0 \leq x \leq 1; \quad f_{Y_n}(x) = 0, \quad x < 0,$ or $x > 1$. Thus,

$$E[e^{itY_n}] = \int_0^1 e^{itx} n(1-x)^{n-1} dx$$

and

$$\begin{aligned}\textstyle\sum_{n=1}^{+\infty} E[e^{itY_n}] P(N=n) &= e^{-\lambda} \textstyle\int_0^1 e^{itx} \sum_{n=1}^{+\infty} n(1-x)^{n-1} \frac{\lambda^n}{n!} dx \\ &= \lambda e^{-\lambda} \textstyle\int_0^1 e^{itx} \sum_{n=1}^{+\infty} \frac{(\lambda(1-x))^{n-1}}{(n-1)!} dx = \lambda e^{-\lambda} \int_0^1 e^{itx} e^{\lambda(1-x)} dx \\ &= \tfrac{\lambda}{it-\lambda} \left(e^{it-\lambda} - 1 \right).\end{aligned}$$

Finally,

$$\varphi_{Y_N}(t) = e^{-\lambda} + \frac{\lambda}{it-\lambda} \left(e^{it-\lambda} - 1 \right).$$

(b) $E[Y_N] = \frac{1}{i} \varphi'_{Y_N}(0) = \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda}$. Therefore, $E[Y_N] \longrightarrow 0$ as $\lambda \to \infty$.

(c) We have

$$\varphi_{\lambda Y_N}(t) = E[e^{it\lambda Y_N}] = \varphi_{Y_N}(\lambda t) = \frac{1}{it-1} \left(e^{\lambda(it-1)} - 1 \right) + e^{-\lambda}.$$

Since $|e^{\lambda(it-1)}| = e^{-\lambda} \to 0$, as $\lambda \to \infty$, it holds that $e^{\lambda(it-1)} \to 0$, as $\lambda \to \infty$. Therefore,

$$\varphi_{\lambda Y_N}(t) \longrightarrow \frac{1}{1-it} = \varphi_{\mathrm{Exp}(1)}(t).$$

Thus, $\lambda Y_N \xrightarrow{d} \mathrm{Exp}(1)$.