HOMEWORK 2 SF 2940

1. MARTINGALE ; IF

i) $E[|X_n|] \leq E[1] = 1 < \infty$

since $0 \leq X_n \leq 1$

ii) X_n is $\mathcal{F}_n$ - MEASURABLE

BY CONSTRUCTION

iii) $E[X_{n+1} | \mathcal{F}_n]$

$= \frac{X_n}{2} P(X_{n+1} = \frac{X_n}{2} | \mathcal{F}_n)$

$+ \frac{(1+X_n)}{2} P(X_{n+1} = \frac{1+X_n}{2} | \mathcal{F}_n)$

$= \frac{X_n}{2}(1 - X_n) + \left(\frac{1+X_n}{2}\right) X_n$

$= X_n$. $\therefore$ $E[X_{n+1} | \mathcal{F}_n] = X_n$

THE MARTINGALE PROPERTY

i) – iii) IMPLY THAT

$\{X_n, F_n\}$

IS A MARTINGALE.

2. $S_N = X_1 + \dots + X_N$

X_i I.I.D. N INDPT OF X_i's

⊛ $\mathrm{COV}(S_N, N) = E[S_N \cdot N] - E[S_N]E[N]$

FIRST

① $E[S_N] = E[N] \cdot E[X]$ BY A RESULT

~~REM~~ IN LN.

② $E[S_N \cdot N] = E[E[S_N \cdot N | N]]$

$= E[N E[S_N | N]]$ (TAKE OUT WHAT IS KNOWN)

$$E[N\,E[S_N|N]] = E[N \cdot H(N)]$$

WHERE $H(N) = E[S_N|N]$

$$E[N\,H(N)] = \sum_{n=0}^{\infty} n\,H(n)\,p_N(n)$$

$$= \sum_{n=0}^{\infty} n\,E[S_N|N=n]\,p_N(n)$$

$$= \sum_{n=0}^{\infty} n\,\underbrace{E[S_n|N=n]}_{\substack{E[S_n],\ \text{AS } X \text{ AND } N \text{ INDEP.}\\ = nE[X]}}\,p_N(n)$$

$$= \sum_{n=0}^{\infty} n \cdot n\,E[X]\,p_N(n)$$

$$= E[X]\,E[N^2]$$

INSERT (1) AND (2) IN (*)

$$\mathrm{Cov}(S_N, N) = E[S_N \cdot N] - E[S_N]E[N]$$

$$= E[X]E[N^2] - E[X](E[N])^2$$

$$= E[X]\left(E[N^2] - (E[N])^2\right)$$

$$= \underline{E[X] \cdot \mathrm{Var}[N]}.$$

3. $X \in U(1, \dots, n)$

$$g_X(t) = \sum_{k=1}^{n} t^k p_X(k)$$

$$= \frac{1}{n} \sum_{k=1}^{n} t^k = \frac{1}{n}\left(\sum_{k=0}^{n} t^k - 1\right)$$

$$= \frac{1}{n}\left(\frac{1 - t^{n+1}}{1-t} - 1\right)$$

$$= \frac{1}{n} \frac{t - t^{n+1}}{1-t}$$

$\lim_{t \to 1} g_X'(t)$ is UNDETERMINED, SINCE

$$g_X'(t) = \frac{1}{n} \frac{(1-(n+1)t^n)(1-t) - (t - t^{n+1})(-1)}{(1-t)^2}$$

$$= \frac{1}{n} \cdot \frac{f(t)}{h(t)}$$

L' HOSPITAL'S RULE

$$g_X'(1) = \frac{1}{n} \lim \frac{f'(t)}{g'(t)}$$

$$f(t) = (1-(n+1)t^n)(1-t) - (t-t^{n+1})(-1)$$

$$f'(t) = -(n+1)n\,t^{n-1}(1-t)$$

$$+ (1-(n+1)t^n)(-1)$$

$$- (1-(n+1)t^n)(-1)$$

$$= -(n+1)n\,t^{n-1} + (n+1)n\,t^n$$

$$-\cancel{1} + \cancel{(n+1)t^n}$$

$$+ \cancel{1} - \cancel{(n+1)t^n}$$

$$= (-1)(n+1)n\,(t^{n-1} - t^n) = (-1)(n+1)\,n\,t^{n-1}(1-t)$$

$$h'(t) = \frac{d}{dt}(1-t)^2 = 2(1-t)(-1)$$

$$\therefore \frac{1}{n}\lim_{t\to 1} \frac{(n+1)n(1-t)}{2(1-t)} = \frac{n+1}{2} \quad \boxed{E[X] = \frac{n+1}{2}}$$

4. a) $\sum_{k=1}^{\infty} P(\underline{Y}=k) = 1$ $\Leftrightarrow$

$$c \sum_{k=1}^{\infty} \frac{1}{k!} = 1 \Rightarrow c = \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k!}}$$

$$\sum_{k=1}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} - 1 = \underline{\underline{e-1}} \qquad \boxed{c = \frac{1}{e-1}}$$

b) $M = \max(U_1, \dots, U_{\underline{Y}})$

$$P(M \leq t) = \sum_{k=1}^{\infty} P(M \leq t, \underline{Y} = k)$$

$$= \sum_{k=1}^{\infty} P(M \leq t \mid \underline{Y} = k) P(\underline{Y} = k)$$

$$= \sum_{k=1}^{\infty} P(\max(U_1, \dots, U_{\underline{Y}}) \leq t \mid \underline{Y} = k) P(\underline{Y} = k)$$

$$= \sum_{k=1}^{\infty} P(\max(U_1, \dots, U_k) \leq t) P(\underline{Y} = k)$$

INDEPENDENCE

$$= \sum_{k=1}^{\infty} P\left(\bigcap_{i=1}^{k} \{U_i \le t\}\right) P(I = k)$$

IND.

$$\overset{\downarrow}{=} \sum_{k=1}^{\infty} \prod_{i=1}^{k} P(U_i \le t) P(I = k)$$

$\forall i$: $P(U_i \le t) = t$, AS $U_i \in U(0,1)$

$$= \sum_{k=1}^{\infty} t^k P(I = k) = g_I(t)$$

BY DEFINITION OF P.G.F

ALSO: $\sum_{k=1}^{\infty} t^k P(I = k) = \frac{1}{(e-1)} \sum_{k=1}^{\infty} \frac{t^k}{k!}$

$$= \frac{1}{(e-1)} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1 \right) = \frac{e^t - 1}{e - 1} = g_I(t)$$

5. $$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$E[S_n] = \frac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n} \cdot n \cdot \mu = \mu$$

$$\therefore \quad E[(S_n - \mu)^2] = Var(S_n)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i)$$ SINCE $Cov(X_i, X_j) = 0$, $i \neq j$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} c$$ BY ASSUMPTION

$$= \frac{c}{n} \to 0, \text{ as } n \to \infty$$

$$\therefore \quad S_n \xrightarrow{2} \mu, \text{ as } n \to \infty.$$

6. $Y \mid X = x \in Po(x)$

$X \in \text{Erlang}(n, 1)$

a) $\varphi_Y(t) = E\left[e^{itY}\right]$

$= E\left[E\left[e^{itY} \mid X\right]\right]$

$= E\left[e^{X(e^{it}-1)}\right]$

Since the ch. fctn of a $Po(x)$ - r.v. is $e^{x(e^{it}-1)}$

$$E\left[e^{X(e^{it}-1)}\right] = \int_0^\infty e^{x(e^{it}-1)} \frac{1}{\Gamma(n)} x^{n-1} e^{-x} \, dx$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-x\underbrace{(1-(e^{it}-1))}_{=(2-e^{it})}} \, dx$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-x/a} \, dx$$

WHERE $a = \dfrac{1}{(2 - e^{it})}$

HENCE

$$\int_0^\infty x^{n-1} e^{-x/a} \, dx = \Gamma(n) \cdot a^n$$

$$\left(\frac{1}{\Gamma(n)} x^{n-1} \frac{1}{a^n} e^{-x/a} \text{ is p.d.f. of } \Gamma(n, a) \right)$$

$$\therefore \varphi_{\Xi}(t) = \frac{1}{\Gamma(n)} \cdot \Gamma(n) \cdot a^n = \underline{\underline{\left(\frac{1}{2 - e^{it}} \right)^n}}$$

NEXT

$$\left(\frac{1}{2 - e^{it}} \right)^n = \left(\frac{\frac{1}{2}}{1 - \frac{e^{it}}{2}} \right)^n \text{ WHICH}$$

IS THE CH. FCTN OF $NBin(n, \frac{1}{2})$.

HENCE $Y \in NBin(n, \frac{1}{2})$

BY UNIQUENESS.

THEN (GUT 8)

$$E[Y] = n$$

$$Var[Y] = 2n$$

THUS

$$\frac{Y - E[Y]}{\sqrt{Var(Y)}} = \frac{Y - n}{\sqrt{2n}}$$

BY EXERCISE 5.8.2

$$Y \overset{d}{=} X_1 + \dots + X_n$$

WHERE X_i I.I.D. $\in Ge(\frac{1}{2})$

$$E[X_i] = 1, \quad Var[X_i] = 2$$

∴ HENCE

$$\frac{Y_n - n}{\sqrt{2n}} = \frac{1}{\sqrt{n}\sqrt{2}} \sum_{i=1}^{n} (x_i - 1)$$

AND BY CLT

$$\frac{1}{\sqrt{n}\sqrt{2}} \sum_{i=1}^{n} (X_i - 1) \to N(0,1)$$

as $n \to \infty$.

7. $X_n(\omega) = n\omega^n \to 0$

as $n \to \infty$

For $0 < \omega < 1$

$X_n(1) = n \to \infty$, as $n \to \infty$

But $P(\{1\}) = 0$

$\therefore X_n \xrightarrow{a.s.} 0$, as $n \to \infty$.

$\therefore X_n \xrightarrow{P} 0$, as $n \to \infty$

$X_n \xrightarrow{d} 0$, as $n \to \infty$

CONVERGENCE IN MEAN SQUARE

IF $X_n \xrightarrow{2} X$, as $n \to \infty$

IT MUST BE THAT $X = 0$.

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CONVERGENCE IN MEAN SQUARE

IF $X_n \xrightarrow{2} X$, as $n \to \infty$

IT MUST BE THAT $X = 0$.

$$E\left[(\overline{X}_n - 0)^2\right] = E\left[\overline{X}_n^2\right] =$$

LAW OF THE UNCONS. STATISTICIAN

$$= \int_0^1 (n\,\omega^n)^2 \, d\omega$$

$\omega \in U(0,1)$

$$= \int_0^1 n^2 \omega^{2n} \, d\omega = \frac{n^2}{2n+1} \to \infty$$

as $n \to \infty$.

$$\therefore \quad \overline{X}_n \not\xrightarrow{2} 0$$

$E\left[(\overline{X}_m - \overline{X}_n)^2\right]$ SHOWS THE SAME THING

$\Downarrow$

CAUCHY CRITERION

8. $X_n = \cos(n\theta)$ $\quad \theta \in U(0, 2\pi)$

1) X_n DOES NOT CONVERGE ALMOST SURELY IN VIEW OF HOW $\cos(x)$ BEHAVES

2) CAUCHY CRITERION

$$E\left[(X_n - X_n)^2\right] \to 0 \text{ ?? as } n \to \infty$$

SET $X_n = \cos(2\pi n \omega)$ $\quad \omega \in U(0,1)$

$$X_{2n} = 2\omega \sin(2\pi n \omega)\cos(2\pi n \omega)$$

$$E\left[(X_n - X_n)^2\right] =$$

$$\int_0^1 \omega^2 \left(\sin(2\pi n \omega)\right)^2 \left(1 - 2\cos(2\pi n \omega)\right) d\omega$$

WHICH DOES NOT CONVERGE TO 0.

HENCE

$$X_n \overset{2}{\not\to} \quad \text{AS} \quad n \to \infty .$$

BUT $X_n = \cos(2\pi n \omega)$

IS BOUNDED

$$P(|X_n| \le 1) = 1$$

HENCE, BY HOMEWORK ASSIGN. 10

IF $X_n \xrightarrow{p} X$, THEN

IT FOLLOWS THAT

$$X_n \xrightarrow{2} X, \quad \text{AS} \quad n \to \infty$$

SINCE WE KNOW THAT

$X_n \overset{2}{\not\to} \cancel{X}$, WE HAVE

THAT X_n DOES NOT

CONVERGE IN PROBABILITY.

$\theta \in U(0, 2\pi)$ $\quad F_\theta(x) = P(\theta \leq x)$

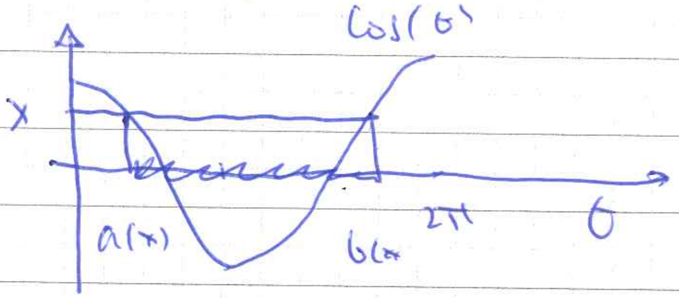
$$= \begin{cases} 0 & x < 0 \\ \frac{x}{2\pi} & 0 < x \leq 2\pi \\ 1 & x \geq 2\pi \end{cases}$$

$$F_{n\theta}(x) = P(n\theta \leq x) = P\left(\theta \leq \frac{x}{n}\right) = F_\theta\left(\frac{x}{n}\right)$$

$$= \frac{x}{2\pi n}$$

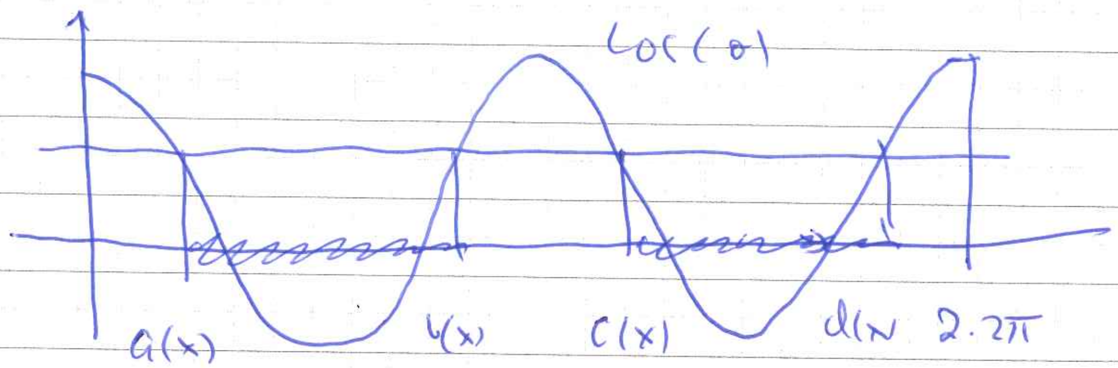
$\therefore$ ~~$n\theta \in U(0, 2\pi n)$~~ $\therefore n\theta \in U(0, 2\pi n)$

$\underline{n = 1}$



$$P(\cos(\theta) \leq x) = P(a(x) \leq \theta \leq b(x)) = \frac{b(x) - a(x)}{2\pi}$$

$n = 2$



$$P(\cos(2\theta) \leq x) = \frac{b(x) - a(x)}{2 \cdot 2\pi} + \frac{d(x) - c(x)}{2 \cdot 2\pi}$$

$a(x) = d(x) - c(x)$

$$= 2 \cdot \frac{b(x) - a(x)}{2 \cdot 2\pi} = P(\cos(\theta) \leq x)$$

9. $Y_1 = X_1 - 4X_2$

$Y_2 = X_1$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= B \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$E[Y_1] = 0 - 4 \cdot 1 = -4$$

$$E[Y_2] = 0.$$

$$Cov_Y = B \, Cov_X \, B^T$$

$$= \begin{pmatrix} 1 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in N\left(\begin{pmatrix} -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 13 & -1 \\ -1 & 1 \end{pmatrix}\right)$$

$$Y_2 \mid Y_1 = 5 \in N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(5+4), \sigma_2^2(1-\rho^2)\right)$$

$$= N\left(0 + \rho \frac{1}{\sqrt{13}} 9, 1 - \rho^2\right)$$

$$\rho = -\frac{1}{\sqrt{13}}$$

$$= N\left(-\frac{9}{13}, 1 - \frac{1}{13}\right) = N\left(-\frac{9}{13}, \frac{12}{13}\right)$$

$$\therefore P(Y_2 \leq 1 \mid Y_1 = 5) = P(Z \leq 1)$$

$$Z \in N\left(-\frac{9}{13}, \frac{12}{13}\right)$$

$$= P\left(\frac{Z + 9/13}{\sqrt{\frac{12}{13}}} \leq \frac{1 + 9/13}{\sqrt{\frac{12}{13}}}\right)$$

$$= \Phi\left(\frac{\frac{22}{13}}{\sqrt{12/13}}\right).$$

10

a) $|\underline{X}| = |\underline{X} - \underline{X}_n + \underline{X}_n|$

$\leq |\underline{X} - \underline{X}_n| + |\underline{X}_n|$

HENCE IF $|\underline{X}| \geq L + \varepsilon$ THEN

$|\underline{X} - \underline{X}_n| + |\underline{X}_n| \geq L + \varepsilon \iff$

~~SINCE~~

$|\underline{X} - \underline{X}_n| \geq L + \varepsilon - |\underline{X}_n|$

SINCE $|\underline{X}_n| \leq L$ $\boxed{\text{W.P. } 1}$

$|\underline{X} - \underline{X}_n| \geq L + \varepsilon - L = \varepsilon$

$\because$ IF $|\underline{X}| \geq L + \varepsilon$ THEN ~~(ALMO~~

$|\underline{X} - \underline{X}_n| \geq \varepsilon$ ~~ALMOST SURELY~~

$\therefore P(|\underline{X}| \geq L + \varepsilon) \leq P(|\underline{X} - \underline{X}_n| \geq \varepsilon)$

$$0 \le P(|\bar{X}| \geq L+\varepsilon) \le P(|\bar{X}_n - \bar{X}| \geq \varepsilon) \to 0 \text{ as } n \to \infty$$

$\therefore P(|\bar{X}| \geq L+\varepsilon) = 0$ FOR ALL $\varepsilon > 0$.

$\therefore P(|\bar{X}| \le L) = 1$.

b) NOTE THAT $0 \le (x+y)^2 = x^2 + 2xy + y^2$

I.E. $-2xy \le x^2 + y^2$

$$0 \le (x-y)^2 = x^2 - 2xy + y^2 \le x^2 + x^2 + y^2 + y^2 = 2x^2 + 2y^2$$

$$\therefore |\bar{X} - \bar{X}_n|^2 \le 2|\bar{X}_n|^2 + 2|\bar{X}|^2$$

$$\le 2L^2 + 2L^2$$ WITH PROB. 1

$$= 4L^2$$

c) $|\underline{X} - \underline{X}_n|^2$

$$= |\underline{X} - \underline{X}_n|^2 \, I_{\{|\underline{X} - \underline{X}_n| \geq \varepsilon\}}$$

$$+ \underbrace{|\underline{X} - \underline{X}_n|^2 \, I_{\{|\underline{X} - \underline{X}_n| < \varepsilon\}}}_{\leq \varepsilon}$$

$$\leq |\underline{X} - \underline{X}_n|^2 \cdot I_{\{|\underline{X} - \underline{X}_n| \geq \varepsilon\}} + \varepsilon$$

Bty b)

$$\leq 4L^2 \, I_{\{|\underline{X} - \underline{X}_n| \geq \varepsilon\}} + \varepsilon$$

$$\therefore E\left[|\underline{X} - \underline{X}_n|^2\right] \leq 4L^2 P(|\underline{X} - \underline{X}_n| \geq \varepsilon) + \varepsilon$$

$\downarrow$

0, as $n \to \infty$

$$\lim_{n \to \infty} E\left[|\underline{X}_n - \underline{X}|^2\right] = 0$$

HENCE WE HAVE SHOWN THAT

IF $P(|X_n| \leq 2)$ THEN

$$X_n \xrightarrow{P} X \iff X_n \xrightarrow{2} X$$

(NOTE THAT

$$X_n \xrightarrow{2} X \Rightarrow X_n \xrightarrow{r} X$$

HOLDS ALWAYS)