KTH Matematik

Avd. Matematisk statistik

KOMPLETTERINGSMODELL FÖR TENTAMEN I SF2940 SANNOLIKHETSTEORI/ A MODEL for A COMPLETING EXAM (from Fx to E) IN SF2940 PROBABILITY THEORY

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You are expected to answer three questions, as picked up for your on the 1st of December 2015 by the examiner (T.K.), from the questions below. Collections of formulas and calculators are neither needed nor permitted. The questions below are basically small drills on the theory presented in course material or old exam questions.

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1. Let $f : \mathbf{R} \to \mathbf{R}$ be a Borel function, and X be a random variable. Let $Y = f(X)$. Prove that Y is a random variable.

2. $(\Omega, \mathcal{F}, P)$ is a probability space. $A \in \mathcal{F}$, $B \in \mathcal{F}$ and $A \subseteq B$. A^c is the complement of A. Which one of the following statements a) -e) is not *necessarily true*? Justify your answer.

   a) $P(A^c) \leq P(B^c)$.
   b) $P(A \mid B) = 1$.
   c) A and B are dependent events.
   d) $P(B \cap A^c) = P(B) - P(A)$..
   f) $P(B \mid A) = 1$.

3. A random variable X has the p.d.f., $\sigma > 0$,

$$f_X(x) = \begin{cases} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   Then we say that $X \in HN(0, \sigma^2)$, and that X is a half-normal (or folded normal) distributed random variable.

   Let $X \in HN(0, \sigma^2)$ and $c > 0$ be a constant. What is the distribution of $\frac{X}{c}$?

4. Consider the joint probability density $f_{X,Y}(x, y)$ for the bivariate r.v. (X, Y) given as

$$f_{X,Y}(x, y) = \begin{cases} 2 & x, y \geq 0, x + y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

   a) Find $E[X]$ and $E[Y]$.
   b) Find $\text{Var}[X]$ and $\text{Var}[Y]$.

5. Let (X, Y) be a continuous bivariate random variable with the joint probability density $f_{X,Y}(x, y)$. Show that

$$E(Y) = E(E(Y \mid X)).$$

6. Derive the probability generating function (p.g.f.) of S_N

$$S_N = X_1 + X_2 + \ldots + X_N.$$

if we know that X_i are independent, non negative, identically distributed integer valued random valued random variables with p.g.f. $g_X(t)$, and N is independent of X_i's, non negative, integer valued random valued random variable with with p.g.f. $g_N(t)$.

7. Let $X_1, X_2, \ldots,$ be I.I.D. random variables with $X_i \in U(0,1)$. Set

$$G_n = \left(X_1 \cdot X_2 \cdots X_n\right)^{1/n},$$

i.e., G_n is the geometric mean of $X_1, X_2, \ldots$ and X_n. Show that

$$G_n \xrightarrow{P} e^{-1} \quad \text{as } n \to \infty.$$

You may find

$$\int_0^x \ln u \, du = x \cdot \ln x - x, \quad x > 0.$$

useful.

8. Let $\mathbf{W} = \{W(t) \mid t \geq 0\}$ be a Wiener process. Then we know that $W(0) = 0$ and that

$$E(W(t)) = 0, \quad E(W(t)W(s)) = \min(t, s),$$

Show that if $0 < s < t$, then

$$E((W(t) - W(s))^2) = t - s.$$

What is the probability distribution of $W(t) - W(s)$?

9. Let X be a random variable with the characteristic function $\varphi(t)$. Show that

$$\varphi(t) \text{ is real} \Leftrightarrow X \stackrel{d}{=} -X.$$

10. Formulate and prove the central limit theorem for I.I.D. random variables $\{X_i\}_{i=1}^{\infty}$ with mean zero and variance σ^2.

11. Give one example of an application of Chebysjev's or Markov's inequality in probability theory (as expounded in the course).

12. Assume that $N = \{N(t) \mid t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$. Let T_1 be the occurrence time of the first event in N.

    Recall that $\{T_1 \geq t\} = \{N(t) = 0\}$ is true and use this to prove that

$$T_1 \in Exp\,(1/\lambda)\,.$$

13. Let $X_1, X_2, \ldots,$ be I.I.D. random variables with $X_i \in N(0,1)$ Show that

$$\frac{X_1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}} \xrightarrow{d} N(0,1),$$

as $n \to +\infty$.

14. Define a martingale, give an example and explain why your example is a martingale.

15. Give two examples of an application of the Borel- Cantelli lemma in probability theory (as expounded in the Course).

16. Formulate the Cramér-Slutzky theorem. You are not expected to prove it.

17. Let $X_1, X_2, \ldots,$ be I.I.D. random variables with $E(X_i) = 0$ and $\mathrm{Var}(X_i) = \sigma^2$. Show that

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=1}^{n} X_i^2}} \xrightarrow{d} N(0,1),$$

as $n \to +\infty$.