EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT

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Sugested solutions

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Problem 1

(a) First of all we have

$$d_0 = 1.$$

In general the price of a coupon bond with maturity time n and cash flows $c_1, \dots, c_n$ has price

$$P = \sum_{k=1}^{n} c_k d_k.$$

It follows that

$$98.00 = 100 \cdot d_1 \quad \Rightarrow \quad d_1 = 0.9800.$$

$$\begin{aligned} 103.80 &= 3.50 \cdot d_1 + 103.50 \cdot d_2 \\ &= 3.50 \cdot 0.9800 + 103.50 \cdot d_2 \\ \Rightarrow d_2 &= 0.9698 \end{aligned}$$

$$\begin{aligned} 107 &= 4.00 \cdot d_1 + 4.00 \cdot d_2 + 104 \cdot d_3 \\ &= 4.00 \cdot 0.9800 + 4.00 \cdot 0.9698 + 104 \cdot d_3 \\ \Rightarrow d_3 &= 0.9539 \end{aligned}$$

To find d_4 we must first find d_5 and then use linear interpolation between d_3 and d_5.

$$94.00 = 100 \cdot d_5 \quad \Rightarrow \quad d_5 = 0.9400$$

Finally

$$d_4 = d_3 + \frac{d_5 - d_3}{5 - 3} \cdot (4 - 3) = 0.9539 + \frac{0.9400 - 0.9539}{2} = 0.9469.$$

(b) The value of the annuity is

$$\begin{aligned} 50 \cdot d_3 + 50 \cdot d_4 + 50 \cdot d_5 &= 50 \cdot 0.9539 + 50 \cdot 0.9469 + 50 \cdot 0.9400 \\ &= 142.04 \end{aligned}$$

(c) We know that

$$G_0^{(2)} = \frac{S_0 - c \cdot d_1}{d_2},$$

where $c = 12$ is the dividend payment and $S_0 = 80$ is the stock price today. Hence

$$G_0^{(2)} = \frac{80 - 12 \cdot 0.98}{0.9698} = 70.365$$

Problem 2

Let N be the number of insured alive at time T. Then $N \sim Bin(n,p)$ and the liability is

$$L = NS_T.$$

(a) We can create any payoff A on the form

$$A = h_0 + hS_T$$

for $(h_0, h) \in \mathbb{R}^2$. The optimal hedge is given by

$$h = \frac{\text{Cov}(L, S_T)}{\text{Var}(S_T)} \text{ and } h_0 = E[L] - hE[S_T].$$

We have

$$\begin{aligned} \text{Cov}(NS_T, S_T) &= E\left[NS_T^2\right] - E[NS_T]E[S_T] \\ &= E[N]\left(E\left[S_T^2\right] - E[S_T]^2\right) \\ &= E[N]\,\text{Var}(S_T). \end{aligned}$$

Hence

$$h = \frac{E[N]\,\text{Var}(S_T)}{\text{Var}(S_T)} = E[N] = np.$$

Furthermore

$$h_0 = E[NS_T] - npE[S_T] = E[N]E[S_T] - npE[S_T] = (np - np)E[S_T] = 0.$$

(b) Let

$$Z = \begin{bmatrix} S_T \\ n - N \end{bmatrix}.$$

Now we can create any payoff on the form

$$A = h_0 + h^T Z$$

for $(h_0, h) \in \mathbb{R}^3$. We know that the optimal hedge is given by

$$h = \Sigma_Z^{-1}\Sigma_{L,Z} \quad \text{and} \quad h_0 = E[L] - h^T E[Z].$$

Now

$$\Sigma_Z = \text{Cov}(Z) = \begin{bmatrix} \text{Var}(S_T) & 0 \\ 0 & \text{Var}(n-N) \end{bmatrix} = \begin{bmatrix} \text{Var}(S_T) & 0 \\ 0 & \text{Var}(N) \end{bmatrix},$$

since S_T and N are uncorrelated. We also have

$$\Sigma_{L,Z} = \begin{bmatrix} \text{Cov}(NS_T, S_T) \\ \text{Cov}(NS_T, n-N) \end{bmatrix} = \begin{bmatrix} \text{Cov}(NS_T, S_T) \\ -\text{Cov}(NS_T, N) \end{bmatrix},$$

where

$$\begin{aligned}\mathrm{Cov}(NS_T, S_T) &= E[NS_T \cdot S_T] - E[NS_T]\,E[S_T] \\ &= E[N]\,E\left[S_T^2\right] - E[N]\,E[S_T]^2 \\ &= E[N]\,\mathrm{Var}(S_T)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(NS_T, N) &= E[NS_T \cdot N] - E[NS_T]\,E[N] \\ &= E[S_T]\,E\left[N^2\right] - E[S_T]\,E[N]^2 \\ &= E[S_T]\,\mathrm{Var}(N).\end{aligned}$$

Since

$$S_T = S_0 e^{\mu T + \sigma\sqrt{T}Z},$$

where $Z \sim N(0,1)$, we have

$$E[S_T] = S_0 e^{\mu T + \frac{\sigma^2 T}{2}}.$$

Hence

$$\begin{aligned}h &= \Sigma_Z^{-1}\Sigma_{L,Z} = \begin{bmatrix} 1/\mathrm{Var}(S_T) & 0 \\ 0 & 1/\mathrm{Var}(N)\end{bmatrix}\begin{bmatrix} E[N]\,\mathrm{Var}(S_T) \\ -E[S_T]\,\mathrm{Var}(N)\end{bmatrix} \\ &= \begin{bmatrix} E[N] \\ -E[S_T]\end{bmatrix} = \begin{bmatrix} np \\ -S_0 e^{\mu T + \frac{\sigma^2 T}{2}}\end{bmatrix}.\end{aligned}$$

To calculate h_0 we need

$$E[Z] = \begin{bmatrix} E[S_T] \\ E[n-N]\end{bmatrix} \quad \text{and} \quad E[L] = E[NS_T] = E[N]\,E[S_T].$$

Hence

$$\begin{aligned}h_0 &= E[L] - h^T E[Z] = E[N]\,E[S_T] - (E[N]\,E[S_T] - E[S_T]\,(n - E[N])) \\ &= (n - E[N])E[S_T] = n(1-p)S_0 e^{\mu T + \frac{\sigma^2 T}{2}}.\end{aligned}$$

Problem 3

(a) When the interest rate is zero we have $X = -L$. We know that the Value-at-Risk is given by

$$\mathrm{VaR}_\alpha(X) = F_L^{-1}(1-\alpha)$$

for any $\alpha \in (0,1)$. In our case

$$F_L(\ell) = \begin{cases} 0 & \text{if} \quad \ell \in (-\infty, 0) \\ 0.75 + 0.25\left(1 - e^{-\ell/\mu}\right) & \text{if} \quad \ell \in [0, \infty)\end{cases}$$

and it follows that the quantile function is

$$F_L^{-1}(u) = \min\{m | F_L(m \geq 0)\} = \begin{cases} 0 & \text{if} \quad u \in (0, 0.75] \\ -\mu \ln(4(1-u)) & \text{if} \quad u \in (0.75, 1). \end{cases}$$

It follows that

$$\mathrm{VaR}_\alpha(X) = \begin{cases} -\mu \ln(4\alpha) & \text{if} \quad \alpha \in (0, 0.25) \\ 0 & \text{if} \quad \alpha \in [0.25, 1). \end{cases}$$

(b) That the insurance company is risk-neutral means that it uses the utility function

$$u(x) = x.$$

Inserting this function into the equation defining the premium we get

$$E[W + \pi - L] = W \quad \Leftrightarrow \quad \pi = E[L].$$

Now

$$E[L] = 0.75 \cdot 0 + 0.25 \cdot \mu = 0.25 \cdot \mu,$$

so

$$\pi = 0.25 \cdot \mu$$

in this case

(c) In this case we have

$$u(x) = -\tau e^{-x/\tau}$$

and the premium satisfies

$$E\left[-\tau e^{-(W+\pi-L)/\tau}\right] = -\tau e^{-W/\tau}.$$

Simplifying we get

$$e^{\pi/\tau} = E\left[e^{L/\tau}\right] \quad \Leftrightarrow \quad \pi = \tau \ln\left(E\left[e^{L/\tau}\right]\right).$$

Now

$$\begin{aligned} E\left[e^{L/\tau}\right] &= \int_{-\infty}^{\infty} e^{x/\tau} dF_L(x) \\ &= 0.75 \cdot e^{0/\tau} + 0.25 \int_0^{\infty} e^{x/\tau} \frac{1}{\mu} e^{-x/\mu} dx \\ &= 0.75 + \frac{0.25}{\mu} \int_0^{\infty} e^{-x(1/\mu - 1/\tau)} dx \\ &= 0.75 + \frac{0.25}{\mu} \left[\frac{1}{1/\tau - 1/\mu} e^{-x(1/\mu - 1/\tau)}\right]_0^{\infty} \\ &= 0.75 + \frac{0.25}{1 - \mu/\tau}, \end{aligned}$$

where we assume that $\tau > \mu$. Finally

$$\pi = \tau \ln \left(0.75 + \frac{0.25}{1 - \mu/\tau} \right).$$

Problem 4

(a) We know that p is a probability density. Hence we must have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} p(x)dx \\ &= \int_{-\infty}^{\infty} e^{\theta x} q(x)dx \\ &= E_Q\left[e^{\theta X}\right] \\ &= \{X \sim N(\mu, \sigma^2)\} \\ &= e^{\theta\mu + \theta^2\sigma^2/2}. \end{aligned}$$

It follows that we must have

$$\theta\mu + \frac{\theta^2\sigma^2}{2} = 0,$$

and since $\theta \neq 0$ we get

$$\theta = -\frac{2\mu}{\sigma^2}.$$

(b) We get

$$\begin{aligned} p(x) &= e^{\theta x} q(x) \\ &= e^{\theta x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2\sigma^2\theta x\right)} \end{aligned}$$

We know from (a) that $\theta = -2\mu/\sigma^2$, and hence

$$-2\sigma^2\theta = -s\sigma^2 \cdot \frac{-2\mu}{\sigma^2} = 4\mu.$$

It follows that

$$p(x) = \frac{1}{2\pi} e^{\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 + 4\mu x)} = \frac{1}{2\pi} e^{\frac{1}{2\sigma^2}(x^2 + 2\mu x + \mu^2)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\mu)^2}{2\sigma^2}},$$

and we see that $X \sim N(-\mu, \sigma^2)$.

(c) We know that the optimal position $h(x)$ is given by

$$h(x) = (u')^{-1} \left(\lambda \frac{q(x)}{p(x)} \right),$$

where λ is the Lagrange multiplier. With $u(x) = \ln x$ we have

$$u'(x) = \frac{1}{x} \quad \text{and} \quad (u')^{-1}(x) = \frac{1}{x}.$$

It follows that

$$h(x) = \frac{1}{\lambda} \cdot \frac{p(x)}{q(x)} = \frac{1}{\lambda} \cdot e^{\theta x} = \frac{1}{\lambda} e^{-2\mu x/\sigma^2}.$$

The budget constraint is (the interest rate is zero, so $B_0 = 1$)

$$V_0 = \int_{-\infty}^{\infty} h(x) q(x) dx = \frac{1}{\lambda} \int_{-\infty}^{\infty} p(x) dx = \frac{1}{\lambda}.$$

Finally, we get

$$h(x) = V_0 e^{-2\mu x/\sigma^2}$$

Problem 5

(a) A coherent risk measure ρ fulfills the following four properties:

- Translation invariance: $\rho(X + cR_0) = \rho(X) - c$ for every $c \in \mathbb{R}$.
- Monotonicity: $X_1 \leq X_2 \;\Rightarrow\; \rho(X_1) \geq \rho(X_2)$.
- Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for every $\lambda \geq 0$.
- Subadditivity: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

(b) We have

$$ES_{0.025}(X) = \frac{1}{0.025} \int_0^{0.025} \mathrm{VaR}_u(X) du.$$

Furthermore

$$\mathrm{VaR}_u(X) = F_L^{-1}(1-u),$$

and since $R_0 = 1$ we have $L = -X$. Now

$$F_L^{-1}(u) = \begin{cases} -10 & \text{if} \quad u \in (0, 0.985] \\ 200 & \text{if} \quad u \in (0.985, 0.995] \\ 1000 & \text{if} \quad u \in (0.995, 1) \end{cases}$$

and

$$\mathrm{VaR}_u(X) = \begin{cases} 1000 & \text{if} \quad u \in (0, 0.005) \\ 200 & \text{if} \quad u \in [0.005, 0.015) \\ -10 & \text{if} \quad u \in [0.015, 1). \end{cases}$$

It follows that

$$\begin{aligned} ES_{0.025}(X) &= \frac{1}{0.025} \left(\int_0^{0.005} 1000\, du + \int_{0.005}^{0.015} 200\, du + \int_{0.015}^{0.025} (-10)\, du \right) \\ &= 40 \cdot \big(0.005 \cdot 1000 + (0.015 - 0.005) \cdot 200 \\ &\quad + (0.025 - 0.015) \cdot (-10)\big) \\ &= 168. \end{aligned}$$