EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT

Date: 2015-10-29, 08:00-13:00

Sugested solutions

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Problem 1

The price P of a coupon bond with yearly coupon C, face value F and discounted using the zero rates $(r_1, \dots, r_T)$ are given by

$$P = \sum_{k=1}^{T} C e^{-r_k k} + F e^{-r_T T}. \qquad (\star)$$

(a) Using Equation $(\star)$ we get

$$\begin{aligned} P_1 &= 5.00 e^{-0.01\cdot 1} + 100 e^{-0.01\cdot 1} = 103.96 \\ P_2 &= 7.00\left(e^{-0.01\cdot 1} + e^{-0.018\cdot 2}\right) + 200\cdot e^{-0.018\cdot 2} = 206.61 \\ P_3 &= 5.50\left(e^{-0.01\cdot 1} + e^{-0.018\cdot 2} + e^{-0.024\cdot 3} + e^{-0.028\cdot 4} + e^{-0.031\cdot 5}\right) + 150\cdot e^{-0.031\cdot 5} \\ &= 153.96 \end{aligned}$$

where P_i is the price of bond i.

(b) In this case we have to add 0.02 to the risk-free zero rates in order to get the correct discount rates. Hence the price of the bond is given by

$$\begin{aligned} P &= 75\,000\Big(e^{-(0.01+0.02)\cdot 1} + e^{-(0.018+0.02)\cdot 2} \\ &\quad + e^{-(0.024+0.02)\cdot 3} + e^{-(0.028+0.02)\cdot 4}\Big) + 5\,000\,000\cdot e^{-(0.028+0.02)\cdot 4} \\ &= 4\,396\,453. \end{aligned}$$

In general the yield-to-maturity (which is the internal rate of return for a bond) r_0 solves the equation

$$P = \sum_{k=1}^{T} C e^{-r_0 k} + F e^{-r_0 T}.$$

With $d = e^{-r_0}$ this equation can be written

$$P = C \sum_{k=1}^{T} d^k + F d^T.$$

With our bond we get

$$4\,396\,453 = 75\,000\left(d + d^2 + d^3 + d^4\right) + 5\,000\,000 d^4.$$

Solving this equation numerically we get the solution

$$d = 0.9533$$

in $(0, 1)$, so the yield-to-maturity is

$$r_0 = -\ln d = 0.0478.$$

(c) The present value of the liability is given by

$$P_L = 10\,000\,000 \cdot e^{-0.024 \cdot 3} = 9\,305\,309$$

and the duration is

$$D_L = 3$$

(since the ony cash flow is at time 3). We have to use Bond 2, and since we should only use one more bond and we must have a long position in both bonds, the second bond must have a duration that is larger than 3. The duration of a general stream of cash flows is given by

$$D = \frac{1}{P} \sum_{k=1}^{T} c_k e^{-r_k k},$$

where P is the value of the stream of cash flows. Using this formula on the three coupon bonds we get the durations

$$\begin{aligned} D_1 &= 1.00 \\ D_2 &= 1.97 \\ D_3 &= 4.66. \end{aligned}$$

Since we need one bond with duration smaller than $D_L = 3$ and one bond with duration lager than $D_L = 3$ in order to have a immunization portfolio with long positions, and we must use Bond 2, we also have to use Bond 3 in our portfolio. Let x and y denote the amount of Bond 2 and Bond 3 that we buy. The immunization conditions can now be written

$$\begin{aligned} P_L &= h_2 P_2 + h_3 P_3 \\ P_L D_L &= h_2 D_2 P_2 + h_3 D_3 P_3 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} 9\,305\,309 &= h_2 \cdot 206.61 + h_3 \cdot 153.96 \\ 27\,915\,927 &= h_2 \cdot 406.29 + h_3 \cdot 716.94, \end{aligned}$$

where h_2 and h_3 are the number of Bond 2 and Bond 3 we buy respectively. The solution is

$$h_2 = 27\,735 \text{ and } h_3 = 23\,220$$

Problem 2

(a) See the book.

(b) We can create any payoff A on the form

$$A = h_0 + h_1 S_T$$

for $(h_0, h_1) \in \mathbb{R}^2$. In general the optimal hedge is given by

$$h_1 = \frac{\mathrm{Cov}(L, S_T)}{\mathrm{Var}(S_T)} \text{ and } h_0 = E\left[L\right] - hE\left[S_T\right].$$

Here

$$L = NP_T.$$

We get

$$\begin{aligned}
h_1 &= \frac{\mathrm{Cov}(NP_T, S_T)}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[NP_T S_T\right] - E\left[NP_T\right] E\left[S_T\right]}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[N\right] E\left[P_T S_T\right] - E\left[N\right] E\left[P_T\right] E\left[S_T\right]}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[N\right] \left(E\left[P_T S_T\right] - E\left[P_T\right] E\left[S_T\right]\right)}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[N\right] \mathrm{Cov}(P_T, S_T)}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[N\right] \mathrm{Corr}(P_T, S_T)\sigma(P_T)\sigma(S_T)}{\mathrm{Var}(S_T)} \\
&= \frac{E\left[N\right] \mathrm{Corr}(P_T, S_T)\sigma(P_T)}{\sigma(S_T)}.
\end{aligned}$$

We have

$$\begin{aligned}
E\left[N\right] &= \lambda \\
\mathrm{Corr}(P_T, S_T) &= \rho,
\end{aligned}$$

and need to calulate the standard deviations. We get

$$\begin{aligned}
\mathrm{Var}(S_T) &= E\left[S_T^2\right] - E\left[S_T\right]^2 \\
&= E\left[S_0^2 e^{2\ln(S_T/S_0)}\right] - \left(S_0 E\left[e^{\ln(S_T/S_0)}\right]\right)^2 \\
&= \left\{\text{If } X \sim N(\mu, \sigma^2) \text{ then } E\left[e^X\right] = e^{\mu + \sigma^2/2}\right\} \\
&= S_0^2 \left(e^{2\mu T + 4\sigma^2 T/2} - e^{2\mu T + 2\sigma^2 T/2}\right) \\
&= S_0^2 e^{2\mu T} \left(e^{2\sigma^2 T} - e^{\sigma^2 T}\right),
\end{aligned}$$

so

$$\sigma(S_T) = S_0 e^{\mu T}\sqrt{e^{2\sigma^2 T} - e^{\sigma^2 T}}.$$

In general, if $X \sim U(a,b)$ then $E[X] = (a+b)/2$ and $\mathrm{Var}(X) = (b-a)^2/12$. It follows that

$$E[P_T/P_0] = \frac{m + e^{\delta T} + m - e^{\delta T}}{2} = m \quad \Rightarrow \quad E[P_T] = P_0 m$$

and

$$\mathrm{Var}(P_T/P_0) = \frac{\left(m + e^{\delta T} - \left(m - e^{\delta T}\right)\right)^2}{12} = \frac{\left(2e^{\delta T}\right)^2}{12} = \frac{e^{2\delta T}}{3}.$$

We get

$$\mathrm{Var}(P_T) = P_0^2 \frac{e^{2\delta T}}{3}$$

and

$$\sigma(P_T) = P_0 \frac{e^{\delta T}}{\sqrt{3}}.$$

This yields

$$h_1 = \frac{\lambda \cdot \rho \cdot P_0 \frac{e^{\delta T}}{\sqrt{3}}}{S_0 e^{\mu T}\sqrt{e^{2\sigma^2 T} - e^{\sigma^2 T}}} = \frac{\lambda \rho P_0 e^{(\delta - \mu - \sigma^2/2)T}}{S_0\sqrt{3\left(e^{\sigma^2 T} - 1\right)}}$$

and

$$\begin{aligned} h_0 &= E[NP_T] - hE[S_T] \\ &= E[N]\,E[P_T] - hE[S_T] \\ &= \lambda \cdot P_0 m - \frac{\lambda \rho P_0 e^{(\delta - \mu - \sigma^2/2)T}}{S_0\sqrt{3\left(e^{\sigma^2 T} - 1\right)}} S_0 e^{(\mu + \sigma^2/2)T} \\ &= \lambda P_0 m - \frac{\lambda P_0 \rho e^{\delta T}}{\sqrt{3\left(e^{\sigma^2 T} - 1\right)}}. \end{aligned}$$

Problem 3

(a) The minimum variance portfolio solves the problem

$$\left[\begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} & w^T \mathbf{1} = 1. \end{array}\right.$$

The Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda(1 - w^T \mathbf{1}),$$

and the first order conditions are

$$\begin{aligned}\nabla L &= \Sigma w - \lambda \mathbf{1} = 0 \\ w^T \mathbf{1} &= 1.\end{aligned}$$

The solution is given by

$$w = \lambda \Sigma^{-1} \mathbf{1}.$$

We insert this in the constraint to get

$$1 = \lambda \mathbf{1} \Sigma^{-1} \mathbf{1},$$

and it follows that

$$w_{\text{MVP}} = \frac{1}{\mathbf{1}\Sigma^{-1}\mathbf{1}} \cdot \Sigma^{-1}\mathbf{1}.$$

Now

$$\Sigma^{-1} = \begin{bmatrix} 12.21 & 4.3956 & 0 \\ 4.3956 & 17.5824 & 0 \\ 0 & 0 & 25.00 \end{bmatrix}.$$

We get

$$\Sigma^{-1}\mathbf{1} = \begin{bmatrix} 16.61 \\ 21.98 \\ 25.00 \end{bmatrix} \text{ and } \mathbf{1}^T \Sigma^{-1} \mathbf{1} = 63.58,$$

so

$$w_{\text{MVP}} = \begin{bmatrix} 0.261 \\ 0.346 \\ 0.393 \end{bmatrix}.$$

(b) The efficient frontier is the curve in the σ-μ-plane starting at the minimum-variance point and then all the points on the minimum-variance set which has an expected return higher than the expected return on the minimum-variance portfolio.

(c) We know that the minimum variance portfolio is on the efficient frontier – it is the efficient portfolio with the smallest expected return. The return on the minimum variance portfolio is here given by

$$\mu_{\text{MVP}} = w_{\text{MVP}}^T \mu = 1.089.$$

To find another efficient portfolio we solve the problem

$$\left[\begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} & w^T \mu = \mu_0 \\ & w^T \mathbf{1} = 1 \end{array} \right.$$

with some $\mu_0 > \mu_{\text{mvp}}$. The Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda_1 (\mu_0 - w^T \mu) + \lambda_2 (1 - w^T \mathbf{1}),$$

and the first order conditions are

$$\begin{aligned}
\nabla L &= \Sigma w - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0 \\
w^T \mu &= \mu_0 \\
w^T \mathbf{1} &= 1
\end{aligned}$$

The optimal portfolio is

$$w = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbf{1},$$

and the multipliers are determined by

$$\begin{bmatrix} \mu^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mathbf{1} \\ \mathbf{1}^T \Sigma^{-1} \mu & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}.$$

Choosing $\mu_0 = 1.10$ we get the multipliers

$$\lambda_1 = 0.2657 \text{ and } \lambda_2 = -0.2736$$

and the solution

$$\begin{aligned}
w &= 0.2657 \begin{bmatrix} 12.21 & 4.3956 & 0 \\ 4.3956 & 17.5824 & 0 \\ 0 & 0 & 25.00 \end{bmatrix} \begin{bmatrix} 1.08 \\ 1.06 \\ 1.12 \end{bmatrix} \\
&\quad + (-0.2736) \begin{bmatrix} 12.21 & 4.3956 & 0 \\ 4.3956 & 17.5824 & 0 \\ 0 & 0 & 25.00 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.199 \\ 0.201 \\ 0.600 \end{bmatrix}.
\end{aligned}$$

This portfolio and the minimum variance portfolio are two examples of efficient portfolios in this market.

Problem 4

(a) The price π of the payoff X is given by

$$\begin{aligned}
\pi &= B_0 E_Q[X] \\
&= B_0 \int_0^\infty \max(x-K,0)\beta e^{-\beta x} dx \\
&= \{\max(x-K,0) = (x-K)I(x \geq K)\} \\
&= B_0\beta \int_K^\infty (x-K)e^{-\beta x} dx \\
&= B_0\beta \left(\int_K^\infty x e^{-\beta x} dx - K \int_K^\infty e^{-\beta x} dx \right) \\
&= \{\text{Integration by parts in the first integral}\} \\
&= B_0\beta \left(\left[-\frac{x}{\beta} e^{-\beta x} \right]_K^\infty + \frac{1}{\beta} \int_K^\infty e^{-\beta x} dx - K \int_K^\infty e^{-\beta x} dx \right) \\
&= B_0\beta \left[\frac{K}{\beta} e^{-\beta K} + \left(\frac{1}{\beta} - K \right) \cdot \frac{1}{\beta} e^{-\beta K} \right] \\
&= \frac{B_0}{\beta} e^{-\beta K}
\end{aligned}$$

(b) In general the coefficient of absolute risk aversion is given by

$$A(x) = -\frac{u''(x)}{u'(x)}.$$

With $u(x) = 2\sqrt{x}$ we have

$$u'(x) = \frac{1}{\sqrt{x}}, \; u''(x) = -\frac{1}{2x\sqrt{x}} \text{ and finally } A(x) = -\frac{-1/(2x\sqrt{x})}{1/\sqrt{x}} = \frac{1}{2x}.$$

(c) We want to find the function $h : \mathbb{R} \to \mathbb{R}$ that solves the problem

$$\left[\begin{array}{ll} \max & E[u(h(X))] \\ \text{s.t.} & B_0 E_Q[h(X)] = V_0. \end{array} \right.$$

The solution is given by

$$h(x) = (u')^{-1}\left(\lambda \frac{q(x)}{p(x)} \right),$$

where λ is the Lagrange multiplier. With $u(x) = \sqrt{x}$ we have

$$u'(x) = \frac{1}{\sqrt{x}} \text{ and } (u')^{-1}(x) = \frac{1}{x^2}.$$

With the given density functions we get

$$h(x) = \frac{1}{\lambda^2} \cdot \left(\frac{p(x)}{q(x)} \right)^2 = \frac{\beta^2 x^2}{\lambda^2}.$$

To find the value of λ we use the budget constraint:

$$\begin{aligned} V_0 &= B_0 \int_{-\infty}^{\infty} h(x)q(x)dx \\ &= B_0 \int_0^{\infty} \frac{\beta^2 x^2}{\lambda^2} \beta e^{-\beta x} dx \\ &= \frac{B_0 \beta^3}{\lambda^2} \int_0^{\infty} x^2 e^{-\beta x} dx \\ &= \left\{ \text{Integration by parts shows that } \int_0^{\infty} x^2 e^{-\beta x} dx = \frac{2}{\beta^3} \right\} \\ &= \frac{2B_0}{\lambda^2}. \end{aligned}$$

Hence

$$\lambda^2 = \frac{2B_0}{V_0},$$

and we get the optimal derivative position

$$h(x) = \frac{V_0 \beta^2 x^2}{2B_0}.$$

Problem 5

(a) We need to check that ρ satisfies the following three properties:

- Translation invariance:

$$\rho(X + cR_0) = \rho(X) - c \text{ for every } c \in \mathbb{R}.$$

- Monotonicity:

$$X_2 \le X_1 \quad \Rightarrow \quad \rho(X_1) \le \rho(X_2).$$

- Convexity:

$$\rho(\lambda X_1 + (1-\lambda)X_2) \le \lambda \rho(X_1) + (1-\lambda)\rho(X_2)$$

  for every $\lambda \in [0,1]$.

Now we insert the entropic risk measure.

- Translation invariance.

$$\begin{aligned} \rho(X + cR_0) &= \frac{\tau}{R_0} \ln E\left[e^{-(X+cR_0)/\tau}\right] \\ &= \frac{\tau}{R_0} \ln E\left[e^{-X/\tau} e^{-cR_0/\tau}\right] \\ &= \frac{\tau}{R_0} \ln E\left[e^{-X}\right] - c \\ &= \rho(X) - c. \end{aligned}$$

- Monotonicity. Take $X_2 \leq X_1$. Then

$$e^{-X_1/\tau} \leq e^{-X_2/\tau}$$

since $x \mapsto e^{-x/\tau}$ is a decreasing function. It follows that

$$E\left[e^{-X_1/\tau}\right] \leq E\left[e^{-X_2/\tau}\right].$$

Since the logarithmic function is increasing the inequality is preserved when we apply it:

$$\ln E\left[e^{-X_1/\tau}\right] \leq \ln E\left[e^{-X_2/\tau}\right].$$

Finally, the inequality is preserved when we multiply it with the strictly positive constant τ/R_0:

$$\frac{\tau}{R_0} \ln E\left[e^{-X_1/\tau}\right] \leq \frac{\tau}{R_0} \ln E\left[e^{-X_2/\tau}\right]$$
$$\Leftrightarrow$$
$$\rho(X_1) \leq \rho(X_2).$$

- Convexity. Take $\lambda \in [0,1]$. The function $x \mapsto e^{-x/\tau}$ is convex, and hence

$$e^{-(\lambda X_1+(1-\lambda)X_2)/\tau} = e^{-\left(\lambda \frac{X_1}{\tau}+(1-\lambda)\frac{X_2}{\tau}\right)} \leq \lambda e^{-X_1/\tau} + (1-\lambda)e^{-X_2/\tau}.$$

It follows that

$$E\left[e^{-(\lambda X_1+(1-\lambda)X_2)/\tau}\right] \leq \lambda E\left[e^{-X_1/\tau}\right] + (1-\lambda)E\left[e^{-X_2/\tau}\right],$$

and, arguing as above,

$$\frac{\tau}{R_0} \ln E\left[e^{-(\lambda X_1+(1-\lambda)X_2)/\tau}\right] \leq \lambda \frac{\tau}{R_0} \ln E\left[e^{-X_1/\tau}\right] + (1-\lambda)\frac{\tau}{R_0} \ln E\left[e^{-X_2/\tau}\right]$$
$$\Leftrightarrow$$
$$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2).$$

We can conlcude that the entropic risk measure is a convex measure of risk.

(b) We have, using the notation from the book,

$$X = V_1 - R_0 V_0,$$

where V_1 is the payoff from the project in one year, $V_0 = 1\,000\,000$ and $R_0 = 1.05$. The discounted loss L is defined by

$$L = -\frac{X}{R_0} = V_0 - \frac{V_1}{R_0},$$

and we know that

$$\mathrm{VaR}_p(X) = F_L^{-1}(1-p).$$

We have

$$L = \begin{cases} -8\,523\,810 & \text{with probability} \quad 0.20 \\ -904\,762 & \text{with probability} \quad 0.75 \\ 1\,000\,000 & \text{with probability} \quad 0.04 \\ 1\,952\,381 & \text{with probability} \quad 0.01 \end{cases}$$

and

$$F_L^{-1}(p) = \begin{cases} -8\,523\,810 & \text{if} \quad p \leq 0.20 \\ -904\,762 & \text{if} \quad 0.20 < p \leq 0.95 \\ 1\,000\,000 & \text{if} \quad p < 0.95 \leq 0.99 \\ 1\,952\,381 & \text{if} \quad p > 0.99 \end{cases}$$

It follows that

(i)

$$\mathrm{VaR}_{0.05}(X) = F_L^{-1}(0.95) = -904\,762.$$

(ii)

$$\mathrm{VaR}_{0.005}(X) = F_L^{-1}(0.995) = 1\,952\,381.$$