EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT

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Suggested solutions

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**Problem 1**

(a) The relation between the price $P$ of a bond with coupon rate $C$, face value $F$, maturity in $n$ years and yield-to-maturity $y$ is

$$P = \sum_{k=1}^{n} CFe^{-yk} + Fe^{-yn}.$$

The formula for using zero rates $r_k$, $k = 1, \dots, n$ when valuing bonds is

$$P = \sum_{k=1}^{n} CFe^{-r_k k} + Fe^{-r_n n}.$$

For Bond 1 we get

$$103e^{-0.035} = 103e^{-r_1} \quad \Rightarrow \quad r_1 = 0.035,$$

for Bond 2 we get

$$4e^{-0.045} + 104e^{-0.045\cdot 2} = 4e^{-0.035} + 104e^{-r_2\cdot 2} \quad \Rightarrow \quad r_2 = 0.0452,$$

and finally for Bond 3 we get

$$6e^{-0.05}+6e^{-0.05\cdot 2}+106e^{-0.05\cdot 3} = 6e^{-0.035}+6e^{-0.0452\cdot 2}+106e^{-r_3\cdot 3} \quad \Rightarrow \quad r_3 = 0.0505$$

(b) The price of the bond is given by

$$P = 11e^{-0.035} + 11e^{-0.0452\cdot 2} + 211e^{-0.0505\cdot 3} = 202.00$$

(c) If the bond defaults, then the present value is equal to

$$0.20 \cdot 100 \cdot e^{-0.035} = 19.31,$$

and if the bond does not default, then the present value is

$$100 \cdot e^{-0.0505\cdot 3} = 85.94.$$

The price $P$ of the bond is thus

$$P = 0.025 \cdot 19.31 + 0.975 \cdot 85.94 = 84.27.$$

## Problem 2

(a) The optimal portfolio is given by

$$\begin{aligned} h &= \Sigma_Z^{-1}\Sigma_{L,Z} \\ h_0 &= E[L] - h^T E[Z]. \end{aligned}$$

Here

$$\Sigma_{L,Z} = \begin{bmatrix} \mathrm{Cov}(L, Z_1) \\ \dots \\ \mathrm{Cov}(L, Z_n) \end{bmatrix}.$$

Using the optimal portfolio we get the optimal payoff:

$$\hat{A} = E[L] - h^T E[Z] + \underbrace{\left(\Sigma_Z^{-1}\Sigma_{L,Z}\right)^T}_{=\Sigma_{L,Z}^T \Sigma_Z^{-1}} Z = E[L] + \Sigma_{L,Z}^T \Sigma_Z^{-1} (Z - E[Z]).$$

Since

$$E\left[\hat{A}\right] = E[L] + \Sigma_{L,Z}^T \Sigma_Z^{-1} (E[Z] - E[Z]) = E[L]$$

we have

$$E\left[(\hat{A} - L)^2\right] = \left(E\left[\hat{A} - L\right]\right)^2 + \mathrm{Var}(\hat{A} - L) = \mathrm{Var}(\hat{A} - L).$$

It follows that

$$\begin{aligned} E\left[(\hat{A} - L)^2\right] &= \mathrm{Var}(\hat{A} - L) \\ &= \mathrm{Var}\left(E[L] + \Sigma_{L,Z}^T \Sigma_Z^{-1} (Z - E[Z]) - L\right) \\ &= \mathrm{Var}\left(\Sigma_{L,Z}^T \Sigma_Z^{-1} Z - L\right) \\ &= \Sigma_{L,Z}^T \Sigma_Z^{-1} \Sigma_Z \Sigma_Z^{-1} \Sigma_{L,Z} - 2\Sigma_{L,Z} \Sigma_Z^{-1} \Sigma_{L,Z} + \mathrm{Var}(L) \\ &= \Sigma_{L,Z}^T \Sigma_Z^{-1} \Sigma_{L,Z} - 2\Sigma_{L,Z}^T \Sigma_Z^{-1} \Sigma_{L,Z} + \mathrm{Var}(L) \\ &= \mathrm{Var}(L) - \Sigma_{L,Z}^T \Sigma_Z^{-1} \Sigma_{L,Z}. \end{aligned}$$

(b) In this case $L = S_T^2$ and we get

$$h = \frac{\mathrm{Cov}(S_T^2, S_T)}{\mathrm{Var}(S_T)} \text{ and } h_0 = E\left[S_T^2\right] - \frac{\mathrm{Cov}(S_T^2, S_T)}{\mathrm{Var}(S_T^2)} \cdot E[S_T].$$

Since

$$\mathrm{Var}(S_T) = E\left[S_T^2\right] - (E[S_T])^2$$

and

$$\mathrm{Cov}(S_T^2, S_T) = E\left[S_T^3\right] - E\left[S_T^2\right] E[S_T]$$

we need $E[S_T^n]$ for $n = 1, 2, 3$. Let $Z \sim N(0, 1)$. Using the fact that

$$E\left[e^{a+bZ}\right] = e^{a+b^2/2}$$

for $a, b \in \mathbb{R}$ we get

$$\begin{aligned} E\left[S_T\right] &= S_0 E\left[e^{\mu T+\sigma\sqrt{T}Z}\right] \\ &= S_0 e^{\mu T+\sigma^2 T/2}, \\ E\left[S_T^2\right] &= S_0^2 E\left[e^{2\mu T+2\sigma\sqrt{T}Z}\right] \\ &= S_0^2 e^{2\mu T+2\sigma^2 T}, \text{ and} \\ E\left[S_T^3\right] &= S_0^2 E\left[e^{3\mu T+3\sigma\sqrt{T}Z}\right] \\ &= S_0^3 e^{3\mu T+9\sigma^2 T/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{Cov}(S_T^2, S_T) &= S_0^3 e^{3\mu T+9\sigma^2 T/2} - S_0^2 e^{2\mu T+2\sigma^2 T} S_0 e^{\mu T+\sigma^2 T/2} \\ &= S_0^3 e^{3\mu T} e^{5\sigma^2 T/2}\left(e^{2\sigma^2 T}-1\right) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Var}(S_T) &= S_0^2 e^{2\mu T+2\sigma^2 T} - \left(S_0 e^{\mu T+\sigma^2 T/2}\right)^2 \\ &= S_0^2 e^{2\mu T} e^{\sigma^2 T}\left(e^{\sigma^2 T}-1\right). \end{aligned}$$

Hence

$$\begin{aligned} h &= \frac{S_0^3 e^{3\mu T} e^{5\sigma^2 T/2}\left(e^{2\sigma^2 T}-1\right)}{S_0^2 e^{2\mu T} e^{\sigma^2 T}\left(e^{\sigma^2 T}-1\right)} \\ &= S_0 e^{\mu T+3\sigma^2 T/2}\left(e^{\sigma^2 T}+1\right) \end{aligned}$$

and

$$\begin{aligned} h_0 &= S_0^2 e^{2\mu T+2\sigma^2 T} - S_0 e^{\mu T+3\sigma^2 T/2}\left(e^{\sigma^2 T}+1\right) S_0 e^{\mu T+\sigma^2 T/2} \\ &= -S_0^2 e^{2\mu T+3\sigma^3 T}. \end{aligned}$$

The minimal expected square heding error is given by

$$\begin{aligned} E\left[(\hat{A}-L)^2\right] &= \operatorname{Var}(L) - \frac{\left(\operatorname{Cov}(L,Z)\right)^2}{\operatorname{Var}(Z)} \\ &= \operatorname{Var}(S_T^2) - \frac{\left(\operatorname{Cov}(S_T^2, S_T)\right)^2}{\operatorname{Var}(S_T)}. \end{aligned}$$

Now

$$\begin{aligned} \operatorname{Var}(S_T^2) &= E\left[S_T^4\right] - \left(E\left[S_T^2\right]\right)^2 \\ &= S_0^4 e^{4\mu T+8\sigma^2 T} - S_0^4 e^{4\mu T+4\sigma^2 T} \\ &= S_0^4 e^{4\mu T+4\sigma^2 T}\left(e^{4\sigma^2 T}-1\right), \end{aligned}$$

so

$$\begin{aligned}
E\left[(\hat{A}-L)^2\right] &= \mathrm{Var}(S_T^2) - \left(\frac{\mathrm{Cov}(S_T^2, S_T)}{\mathrm{Var}(S_T)}\right)^2 \mathrm{Var}(S_T) \\
&= S_0^4 e^{4\mu T + 4\sigma^2 T}\left(e^{4\sigma^2 T} - 1\right) - \\
&\quad \left(S_0 e^{\mu T + 3\sigma^2 T/2}\left(e^{\sigma^2 T} + 1\right)\right)^2 S_0^2 e^{2\mu T} e^{\sigma^2 T}\left(e^{\sigma^2 T} - 1\right) \\
&= S_0^4 e^{4\mu T + 5\sigma^2 T}\left(e^{2\sigma^2 T} - 1\right)\cdot\left(e^{\sigma^2 T} - 1\right).
\end{aligned}$$

## Problem 3

The inverse of the covariance matrix is

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \dots & 0 & 0 \\ 0 & 1/\sigma_2^2 & 0 & \dots & 0 \\ \dots & & \ddots & & \dots \\ 0 & 0 & \dots & 0 & 1/\sigma_n^2 \end{bmatrix}.$$

(a) The minimum-variance portfolio solves the problem

$$\left[\begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{subject to} & w^T \mathbf{1} = 1. \end{array}\right.$$

The Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda(1 - w^T \mathbf{1}),$$

and the first order condition is

$$\Sigma w - \lambda \mathbf{1} = 0.$$

The solution is

$$w = \lambda \Sigma^{-1} \mathbf{1} = \lambda \begin{bmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ \vdots \\ 1/\sigma_n^2 \end{bmatrix}.$$

Inserting this in the constraint yields

$$\lambda \mathbf{1}^T \Sigma \mathbf{1} = 1 \quad \Rightarrow \quad \lambda = \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

Now

$$\mathbf{1}^T \Sigma^{-1} \mathbf{1} = [1 \ 1 \ \cdots \ 1] \begin{bmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ \vdots \\ 1/\sigma_n^2 \end{bmatrix} = \sum_{k=1}^{n} \frac{1}{\sigma_k^2}.$$

It follows that

$$w_{\text{mvp}} = \frac{1}{\sum_{k=1}^{n} \frac{1}{\sigma_k^2}} \begin{bmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ \vdots \\ 1/\sigma_n^2 \end{bmatrix}.$$

(b) The problem now is

$$\left[ \begin{array}{ll} \max & w^T\mu - \frac{\gamma}{2} w^T \Sigma w \\ \text{subject to} & w^T \mathbf{1} = 1. \end{array} \right.$$

The Lagrangian is

$$L = w^T\mu - \frac{\gamma}{2} w^T \Sigma w + \lambda(1 - w^T \mathbf{1}),$$

and the first order condition is

$$\mu - \gamma \Sigma w - \lambda \mathbf{1} = 0.$$

We get

$$w = \frac{1}{\gamma} \left( \Sigma^{-1}\mu - \lambda \Sigma^{-1} \mathbf{1} \right),$$

and using the constraint we arrive at

$$\frac{1}{\gamma} \left( \mathbf{1}^T \Sigma^{-1} \mu - \lambda \mathbf{1}^T \Sigma^{-1} \mathbf{1} \right) = 1 \quad \Rightarrow \quad \lambda = \frac{\mathbf{1}^T \Sigma^{-1} \mu - \gamma}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

We know the expression for $\mathbf{1}^T \Sigma^{-1} \mathbf{1}$ from above, and get

$$\mathbf{1}^T \Sigma^{-1} \mu = \mu^T \Sigma^{-1} \mathbf{1} = [\mu_1 \ \mu_2 \ \cdots \ \mu_n] \begin{bmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ \vdots \\ 1/\sigma_n^2 \end{bmatrix} = \sum_{k=1}^{n} \frac{\mu_k}{\sigma_k^2}.$$

Hence,

$$\lambda = \frac{\sum_{k=1}^{n} \frac{\mu_k}{\sigma_k^2} - \gamma}{\sum_{k=1}^{n} \frac{1}{\sigma_k^2}}$$

and the optimal portfolio for any $\gamma > 0$ is given by

$$w_i = \frac{1}{\gamma \sigma_i^2} \left( \mu_i + \frac{\gamma - \sum_{k=1}^{n} \frac{\mu_k}{\sigma_k^2}}{\sum_{k=1}^{n} \frac{1}{\sigma_k^2}} \right).$$

(c) The problem we want to solve is

$$\left[ \begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{subject to} & w^T \mu = \mu_0 \\ & w^T \mathbf{1} = 1. \end{array} \right.$$

The Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda_1(\mu_0 - w^T \mu) + \lambda_2(1 - w^T \mathbf{1}),$$

and the first order condition is

$$\Sigma w - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0.$$

The solution is given by

$$w = \Sigma^{-1}(\lambda_1 \mu + \lambda_2 \mathbf{1}) = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbf{1}.$$

Inserting this into the constraints we get

$$\begin{aligned} \mu_0 &= \lambda_1 \mu^T \Sigma^{-1} \mu + \lambda_2 \mu^T \Sigma^{-1} \mathbf{1} \\ 1 &= \lambda_1 \mathbf{1}^T \Sigma^{-1} \mu + \lambda_2 \mathbf{1}^T \Sigma^{-1} \mathbf{1}. \end{aligned}$$

In our case

$$\begin{aligned} \mu^T \Sigma^{-1} \mu &= \sum_{k=1}^{n} \frac{\mu_k^2}{\sigma_k^2} =: a \\ \mathbf{1}^T \Sigma^{-1} \mu &= \mu^T \Sigma^{-1} \mathbf{1} = \sum_{k=1}^{n} \frac{\mu_k}{\sigma_k^2} =: b \\ \mathbf{1}^T \Sigma^{-1} \mathbf{1} &= \sum_{k=1}^{n} \frac{1}{\sigma_k^2} =: c. \end{aligned}$$

Then

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix} = \frac{1}{ac - b^2} \begin{bmatrix} c\mu_0 - b \\ a - b\mu_0 \end{bmatrix}.$$

Hence for $i = 1, 2, \ldots, n$ we have the optimal weights

$$\begin{aligned} w_i &= \lambda_1 \frac{\mu_i}{\sigma_i^2} + \lambda_2 \frac{1}{\sigma_i^2} \\ &= \frac{\left(\mu_0 \sum_{k=1}^n \frac{1}{\sigma_k^2} - \sum_{k=1}^n \frac{\mu_k}{\sigma_k^2}\right) \frac{\mu_i}{\sigma_i^2} + \left(\sum_{k=1}^n \frac{\mu_k^2}{\sigma_k^2} - \mu_0 \sum_{k=1}^n \frac{\mu_k}{\sigma_k^2}\right) \frac{1}{\sigma_i^2}}{\sum_{k=1}^n \frac{1}{\sigma_k^2} \cdot \sum_{k=1}^n \frac{\mu_k^2}{\sigma_k^2} - \left(\sum_{k=1}^n \frac{\mu_k}{\sigma_k^2}\right)^2}. \end{aligned}$$

**Problem 4**

(a) With $u(x) = \ln(x + 10)$ we get the coefficient of absolute risk aversion

$$A(x) = -\frac{u''(x)}{u'(x)} = -\frac{-\frac{1}{(x+10)^2}}{\frac{1}{x+10}} = \frac{1}{x + 10}.$$

(b) The general problem we want to solve is

$$\left[\begin{array}{ll} \max & \sum_{k=1}^{n} p_k \ln(w_k\theta_k + m) \\ \text{subject to} & \sum_{k=1}^{n} w_k = V_0, \end{array}\right.$$

where $n = 5$, for $k = 1, \ldots, 5$ we have introduced $p_k$ and $\theta_k$ as the probabilities and odds respectivly, $m = 10$ and $V_0 = 100$. We get the Lagrangian

$$L = \sum_{k=1}^{n} p_k \ln(w_k\theta_k + m) + \lambda\left(V_0 - \sum_{k=1}^{n} w_k\right),$$

and the first order conditions

$$\begin{aligned} \frac{p_k\theta_k}{w_k\theta_k + m} - \lambda &= 0, \ k = 1, \ldots, n \\ \sum_{k=1}^{n} w_k &= V_0. \end{aligned}$$

The first set of equations yields

$$w_k = \frac{p_k}{\lambda} - \frac{m}{\theta_k}, \ k = 1, \ldots, n,$$

and inserting this into the constraint yields

$$V_0 = \sum_{k=1}^{n}\left(\frac{p_k}{\lambda} - \frac{m}{\theta_k}\right) = \frac{1}{\lambda} - m\sum_{k=1}^{n}\frac{1}{\theta_k}.$$

Hence, the optimal solution is

$$w_k = V_0 p_k + m\left(p_k\sum_{\ell=1}^{n}\frac{1}{\theta_\ell} - \frac{1}{\theta_k}\right), \ k = 1, \ldots, n.$$

Inserting the values given we get

$$w_1 = 24.95, \ w_2 = 14.84, \ w_3 = 8.83, \ w_4 = 46.86 \text{ and } w_5 = 4.52.$$

(c) The certainty equivalent $C$ solves

$$E\left[u(V_1)\right] = u(C),$$

where $u(x) = \ln(x+10)$. Now the value of $V_1$ in state $k$, which we denote $(V_1)_k$, is given by

$$(V_1)_k = w_k\theta_k.$$

It follows that

$$\begin{aligned} E[u(V_1)] &= \sum_{k=1}^{n} p_k \ln((V_1)_k + 10) \\ &= 0.25 \ln(24.95 \cdot 3.75 + 10) + 0.15 \ln(14.84 \cdot 5.75 + 10) \\ &\quad +0.10 \ln(8.83 \cdot 4.5 + 10) + 0.45 \ln(46.86 \cdot 3.5 + 10) \\ &\quad +0.05 \ln(4.52 \cdot 10 + 10) \\ &= 4.7565. \end{aligned}$$

Hence, the certainty equivalent $C$ solves

$$\ln(C + 10) = 4.7565 \quad \Rightarrow \quad C = 106.33.$$

The absolute risk premium $\pi$ of $V_1$ is defined by

$$\begin{aligned} \pi &= E[V_1] - C \\ &= 0.25 \cdot 24.95 \cdot 3.75 + 0.15 \cdot 14.84 \cdot 5.75 + 0.10 \cdot 8.83 \cdot 4.5 \\ &\quad +0.45 \cdot 46.86 \cdot 3.5 + 0.05 \cdot 4.52 \cdot 10 - 4.7565 \\ &= 116.23 - 4.7565 \\ &= 9.90. \end{aligned}$$

**Problem 5**

We want to calculate the value-at-risk and expected shortfall for the random variable

$$X = V_1 - R_0 V_0.$$

(a) We know that with for $p \in (0,1)$ we have

$$\begin{aligned} \mathrm{VaR}_p(X) &= \min\{m | P(mR_0 + X < 0) \le p\} \\ &= \min\{m | P(mR_0 + V_1 - R_0 V_0 < 0) \le p\} \\ &= \min\{m | P(V_1 < R_0(V_0 - m)) \le p\}. \end{aligned}$$

Since $V_1$ has a continuous distribution we have

$$P(V_1 \le R_0(V_0 - \mathrm{VaR}_p(X))) = p.$$

Now

$$F_{V_1}(x) = \int_0^x f_{V_1}(t) dt = \left[ -e^{-x/a} \right]_0^x = 1 - e^{-x/a},$$

and it follows that

$$1 - e^{-R_0(V_0 - \mathrm{VaR}_p(X))/a} = p \quad \Rightarrow \quad \mathrm{VaR}_p(X) = V_0 + \frac{a}{R_0} \ln(1-p).$$

Inserting the parameter values and using $R_0 = 1/B_0$ we get for $p \in (0,1)$

$$\begin{aligned} \mathrm{VaR}_p(X) &= 100\,000 + 150\,000 \cdot 0.96 \ln(1-p) \\ &= 100\,000 + 144\,000 \ln(1-p). \end{aligned}$$

(b) The expected shortfall of $X$ at level $p \in (0,1)$ is given by

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)du \\
&= \frac{1}{p}\int_0^p (100\,000 + 144\,000\ln(1-u))du \\
&= 100\,000 + 144\,000 \cdot \frac{1}{p}\int_0^p \ln(1-u)du \\
&= 100\,000 + 144\,000 \cdot \frac{1}{p}\int_{1-p}^1 \ln v dv \\
&= 100\,000 + 144\,000 \cdot \frac{1}{p}\left[v\ln v - v\right]_{1-p}^1 \\
&= 100\,000 + 144\,000 \cdot \frac{1}{p}\Big(0 - 1 - \big((1-p)\ln(1-p) - (1-p)\big)\Big) \\
&= 100\,000 + 144\,000 \frac{-(1-p)\ln(1-p) - p}{p} \\
&= -44\,000 + 144\,000\left(\frac{1}{p} - 1\right)\ln\frac{1}{1-p}.
\end{aligned}$$