A short note on American option prices

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1 The set-up

An American call option with strike price K written on some stock gives the holder the right to buy a share of the stock (exercise the option) at the strike price K at any time before and including the time of maturity T. The value of exercising the option at time $t \in [0, T]$ is $\max(S_t - K, 0)$, where S_t is the share price at time t. Similarly for an American put option.

American options are not necessarily stock option. Other examples include American futures options where the underlying price process is the process of futures prices on some asset or good.

Write $c_t^A = c_t^A(S_t, K, t, T)$ and c_t^E for the option price at time t of an American and European call option, respectively. Write p_t^A and p_t^E for the option price at time t of an American and European put option, respectively. Since the cash flow of an American option held to maturity is the cash flow of the corresponding European option, $c_t^A \geq c_t^E$ and $p_t^A \geq p_t^E$ must hold in order not to violate the Law of One Price.

In what follows we assume that we can take long and short positions of arbitrary sizes, that the bid and ask prices coincide, and that short-selling does not lead to additional costs (fees, commissions, etc). We assume that zero-coupon bonds of all maturities are available and that the zero rates are strictly positive. We also assume that we can trade in the underlying asset on which the options are written. The stock is said to be a pure investment asset if it does not pay dividends or give other benefits before time T.

This short note is intended as a complement to the lecture notes [**?**] used in the course SF2701 Financial Mathematics at KTH. More details and further properties of American options can be found in [**?**] on which this note is based. The notation used follows that in [**?**].

2 No-arbitrage relations

Theorem 1. *If* $T_1 < T_2$, *then* $c_t^A(S_t, K, t, T_1) \leq c_t^A(S_t, K, t, T_2)$ *for* $t \in [0, T_1]$.

Proof. Suppose that the inequality does not hold and consider the following strategy. At time t buy the (cheaper) call option with maturity T_2 and short-sell the (more expensive) call option with maturity T_1. This gives a strictly positive cash flow at time t. Whenever the short-sold call option is exercised, exercise the call option maturing at T_2. This produces a zero cash flow at the exercise time of the short-sold call option and cancels the call option position. The strategy violates the Law of One Price since it produces a strictly positive cash flow at time t and no other cash flows. □

Theorem 2. *If the stock is a pure investment asset, then* $c_t^E \geq \max(S_t - Z_{t,T}K, 0)$ *for* $t \in [0, T]$.

Proof. Consider the following strategy: at time t buy the call option, buy K zero-coupon bonds maturing at T with face value 1, short-sell the underlying asset and close the short position at time T.

The payoff at time T is $\max(S_T - K, 0) + K - S_T = \max(S_T, K) - S_T \geq 0$. The initial cash flow is $-c_t^E - Z_{t,T}K + S_t$. Since the payoff is nonnegative, the initial cash flow must be nonpositive in order not to violate the Law of One Price. Similarly, $c_t^E \geq 0$. Therefore, $c_t^E \geq \max(S_t - Z_{t,T}K, 0)$. □

Theorem 3. *If the stock is a pure investment asset, then an American call option is not exercised prior to maturity and* $c_t^A = c_t^E$ *for* $t \in [0, T]$.

Proof. The value of exercising the American call option at time t is $\max(S_t - K, 0)$. Moreover, $c_t^A \geq c_t^E$. The call option is only exercised if $S_t > K$ and in this case Theorem **??** gives

$$c_t^A \geq c_t^E \geq \max(S_t - Z_{t,T}K, 0) > \max(S_t - K, 0), \quad t < T.$$

In particular, at time $t < T$ selling the option is always better for the holder than exercising it. Since the cash flow of an American call option held to maturity is identical to a European call option the Law of One Price implies that the option prices must coincide. □

Theorem 4. *The American call option price* $c_t^A(S_t, K, t, T)$ *is a convex function of* K*, i.e. if* $\lambda \in [0, 1]$*,* $K_1 < K_2$*, and* $K_3 = \lambda K_1 + (1 - \lambda)K_2$*, then*

$$c_t^A(S_t, K_3, t, T) \leq \lambda c_t^A(S_t, K_1, t, T) + (1 - \lambda)c_t^A(S_t, K_2, t, T).$$

Proof. Suppose that the inequality does not hold and consider the following strategy. At time t buy λ call options with strike price K_1, buy $1 - \lambda$ call options with strike price K_2, and short-sell one call option with strike price K_3. This gives a strictly positive cash flow at time t. Whenever the short-sold call option is exercised, exercise the other two call options. This produces the cash flow

$$\begin{aligned} C = {} & \lambda \max(S - K_1, 0) + (1 - \lambda) \max(S - K_2, 0) \\ & - \max(S - \lambda K_1 - (1 - \lambda)K_2, 0) \end{aligned}$$

at the exercise time of the short-sold call option and cancels the call option position, where S denotes the share price at the exercise time. Since $\max(x, 0)$ is a convex function,

$$\begin{aligned}\max(S - \lambda K_1 - (1-\lambda)K_2, 0) &= \max(\lambda(S - K_1) + (1-\lambda)(S - K_2), 0) \\ &\le \lambda \max(S - K_1, 0) + (1-\lambda)\max(S - K_2, 0).\end{aligned}$$

Therefore $C \geq 0$ so the strategy produces a strictly positive cash flow at time t, a nonnegative cash flow at the exercise time of the short-sold call option, and no other cash flows. The strategy thereby violates the Law of One Price. $\square$

Theorem 5. *$c_t^A - p_t^A \le S_t - Z_{t,T}K$ for all $t \in [0, T]$.*

Proof. Consider the following portfolio. At time t buy a put option with strike K and maturity T, short-sell a call option with the same strike and maturity, buy a share of the stock, and short-sell K zero-coupon bonds maturing at T with face value one. Consider the following strategy. If the call option is held to maturity, then we hold the put option to maturity and at that time sell the share. The net payoff in this case is zero. The value of the portfolio excluding the short position in the call option at time u prior to maturity is

$$p_u^A + S_u - Z_{u,T}K = p_u^A + K(1 - Z_{u,T}) + S_u - K > S_u - K$$

which is strictly greater than the exercise value for the call option at that time. If the call option is exercised prior to maturity, then we pay the call option payoff, sell the put option and the share, and close out the short position in the zero-coupon bonds by buying bonds. The net cash flow is in that case positive. The cash flow of the strategy is therefore always nonnegative for the holder of the portfolio. In order not to violate the Law of One Price the cost for buying the portfolio must therefore by nonnegative, i.e. $p_t^A - c_t^A + S_t - Z_{t,T}K \ge 0$. $\square$

References

[1] Harald Lang, *Lectures on Financial Mathematics*, Lecture notes, KTH Mathematics, 2009.

[2] Robert Merton (1973), Theory of Rational Option Pricing, *The Bell Journal of Economics and Management Science*, 4, 141-183.