KTH Matematik

SOLUTION TO EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT 2011-10-18.

Problem 1

In immunization a liability such as a deterministic cash flow is hedged by a bond portfolio. The bond portfolio is selected such that its value equals the value of the liability and such that the resulting total portfolio, assets minus liabilities, is insensitive to a number of scenarios $\Delta\mathbf{r}_1, \dots, \Delta\mathbf{r}_q$ of zero rate changes. Principal Component Analysis (PCA) can be used to decide upon reasonable scenarios to protect the portfolio against.

To perform PCA we need historical zero rate changes and an estimate of the covariance matrix $\mathrm{Cov}(\Delta\mathbf{r})$ from the data. The covariance matrix may be expressed as the product (singluar value decomposition) $\mathrm{Cov}(\Delta\mathbf{r}) = \mathbf{ODO}^{\mathrm{T}}$, where $\mathbf{D}$ is a diagonal matrix with the (strictly positive) eigenvalues $\lambda_1, \dots, \lambda_n$ of $\mathrm{Cov}(\Delta\mathbf{r})$ as diagonal elements and $\mathbf{O}$ is an orthogonal matrix whose columns $\mathbf{o}_1, \dots, \mathbf{o}_n$ are eigenvectors of $\mathrm{Cov}(\Delta\mathbf{r})$, orthogonal, and of length one. The columns of $\mathbf{D}$ and $\mathbf{O}$ can be ordered so that the diagonal elements in $\mathbf{D}$ appear in descending order.The first few eigenvectors $\mathbf{o}_1, \mathbf{o}_2, \dots$ are natural candidates as scenarios to use in the immunization procedure. Indeed, any scenario $\Delta\mathbf{r}$ can be expressed in the orthonormal basis $\{\mathbf{o}_1, \dots, \mathbf{o}_n\}$ and since the eigenvalues $\lambda_1, \lambda_2, \dots$ are descending the error from ignoring the eigenvectors with small eigenvalue will be small.

A criteria for selecting the number of principal components to use is to study

$$\sum_{k=1}^{j} \lambda_k \Big/ \sum_{k=1}^{n} \lambda_k$$

as a function of j. Then we often find that its value for $j = 1$ is close to one. This means that variability of $\Delta\mathbf{r}$ is mainly in the direction of the first principal component $\mathbf{o}_1$. To improve the accuracy of the approximation we may pick a small j, such as $j = 2$ or 3 and use the corresponding eigenvectors as the scenarios in the immunization procedure. For zero rates the first three components can often be interpreted as a parallel shift, a change in slope, and a change in curvature of the zero rate curve.

Suppose we use 3 principal components $\mathbf{o}_1, \mathbf{o}_2$, and $\mathbf{o}_3$. If $P(\mathbf{r})$ represent the value of the liability and $P_k(\mathbf{r})$ is the value of the bonds in the hedging portfolio, then a good choice a hedging portfolio is obtained by solving the system of equations

$$\sum_{k=1}^{m} h_k P_k(\mathbf{r}) = P(\mathbf{r}),$$
$$\sum_{k=1}^{m} h_k \nabla P_k(\mathbf{r})^{\mathrm{T}} \mathbf{o}_l = \nabla P(\mathbf{r})^{\mathrm{T}} \mathbf{o}_l \quad \text{for } l = 1, \dots, 3.$$

Problem 2

Consider first an investment problem without a risk free asset. Suppose the vector of returns of the risky assets has mean $\boldsymbol{\mu}$ and covariance matrix Σ and $\mathbf{w}$ is the vector of amounts invested in each asset. The trade-off problem is to

$$\begin{array}{rl} \text{maximize} & \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} - \frac{c}{2V_0}\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}, \\ \text{subject to} & \mathbf{w}^{\mathrm{T}}\mathbf{1} \leq V_0. \end{array}$$

For each $c > 0$, let $(\sigma(c), \mu(c))$ be the mean and standard deviation of the portfolio that solves the trade-off problem with trade-off parameter c. The efficient frontier is the set of all such pairs.
The efficient frontier is determined by finding, for each $c > 0$, the portfolio $\mathbf{w}_c$ that solves the trade-off problem and then compute

$$\mu(c) = \mathbf{w}_c^{\mathrm{T}}\boldsymbol{\mu},$$
$$\sigma(c) = \sqrt{\mathbf{w}_c^{\mathrm{T}}\Sigma\mathbf{w}_c}.$$

The efficient frontier may look like the solid line in Figure 1.
When there is a risk free asset with return R_0 the trade-off problem is

$$\begin{array}{rl} \text{maximize} & w_0 R_0 + \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} - \frac{c}{2V_0}\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}, \\ \text{subject to} & \mathbf{w}^{\mathrm{T}}\mathbf{1} \leq V_0. \end{array}$$

The efficient frontier is determined in a similar way as just described, and it turns out that it is a straight line. It may look like the dashed line in Figure 1.

Problem 3

Let time 1 be in six months from now, $K = 10^4$ be the payoff of each contract, q_j, $j = 1, \dots, 4$, be the price of each contract, and p_j, $j = 1, \dots, 4$, be the probability of each outcome. The initial capital is $V_0 = 10^4$.
Investing the amounts $\mathbf{w}^{\mathrm{T}} = (w_1, w_2, w_3, w_4)$ in the four contracts gives the value V_1 at time 1 as

$$V_1 = \begin{cases} (K/q_1)w_1, & \text{if "Excellent"}, \\ (K/q_2)w_2, & \text{if "Good"}, \\ (K/q_3)w_3, & \text{if "Poor"}, \\ (K/q_4)w_4, & \text{if "Default"}. \end{cases}$$

The expected utility is then given by

$$E[u(V_1)] = \sum_{j=1}^{4} u\Big(\frac{K}{q_j} w_j\Big) p_j.$$

We want to maximize $E[u(V_1)]$ subject to $\mathbf{w}^{\mathrm{T}}\mathbf{1} \leq V_0$.
The sufficient conditions for optimality are:

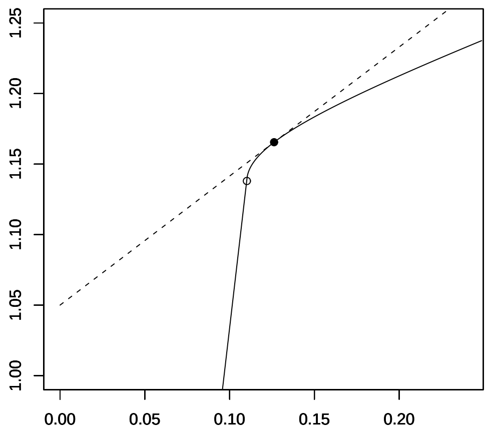

Figure 1: Illustration of the efficient frontier.

(1) $-u'(Kw_j/q_j)p_j + \lambda = 0$, for $j = 1, \dots, 4$,

(2) $\mathbf{w}^{\mathrm{T}}\mathbf{1} \le V_0$,

(3) $\lambda \ge 0$,

(4) $\lambda(\mathbf{w}^{\mathrm{T}}\mathbf{1} - V_0) = 0$.

Since $\tau = 0$ we have $u'(x) = (\gamma x)^{-1/\gamma}$. Assuming $\lambda > 0$, the first condition leads to

$$w_j = \frac{q_j}{\gamma K}\left(\frac{p_j}{\lambda q_j}\right)^{\gamma}.$$

The fourth condition then implies

$$V_0 = \mathbf{w}^{\mathrm{T}}\mathbf{1} = \frac{1}{\lambda^{\gamma}\gamma K}\sum_{j=1}^{4} q_j\left(\frac{p_j}{q_j}\right)^{\gamma},$$

and from this expression we solve for λ to get

$$\lambda = \left(\frac{\sum_{j=1}^{4} q_j\left(\frac{p_j}{q_j}\right)^{\gamma}}{\gamma V_0 K}\right)^{1/\gamma}.$$

Then the optimal portfolio is given by

$$w_j = V_0\frac{q_j\left(\frac{p_j}{q_j}\right)^{\gamma}}{\sum_{k=1}^{4} q_k\left(\frac{p_k}{q_k}\right)^{\gamma}}$$

Entering the numerical values and $\gamma = 2.5$ gives

$$\begin{aligned} w_1 &= 1014.68, \\ w_2 &= 7742.66, \\ w_3 &= 963.76, \\ w_4 &= 278.89. \end{aligned}$$

Problem 4

(a) Let I_1 and I_2 be the default indicators for the two issuers. They are assumed to be independent and identically distributed, $I_1 = 1$ with probability p and $I_1 = 0$ with probability $1 - p$.
The returns of the two bonds are then given by

$$R_k = \frac{10^5}{P_k}(1 - I_k) = \frac{R_0}{1-q}(1 - I_k), \quad k = 1, 2.$$

The expected value of R_k is

$$\mu = E[R_k] = \frac{R_0}{1-q}(1 - E[I_k]) = \frac{R_0}{1-q}(1 - p) = 1.051, \quad k = 1, 2,$$

and the variance is

$$\sigma^2 = V(R_k) = V\Big(\frac{R_0}{1-q}(1 - I_k)\Big) = \frac{R_0^2}{(1-q)^2} p(1-p) = 0.02717.$$

Since the default indicators are independent, so are the returns R_1 and R_2, which implies $\mathrm{Cov}(R_1, R_2) = 0$.
Let $\boldsymbol{\mu}^{\mathrm{T}} = (\mu, \mu)$ be the mean vector of $(R_1, R_2)^{\mathrm{T}}$ and Σ be the covariance matrix given by

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}.$$

Let w_0 be the amount invested in the risk-free bond and $\mathbf{w}^{\mathrm{T}} = (w_1, w_2)$ the amounts invested in the two defaultable bonds, respectively. The objective is to solve

$$\begin{array}{rl} \text{maximize} & w_0 R_0 + \mathbf{w}^{\mathrm{T}} \boldsymbol{\mu}, \\ \text{subject to} & \mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w} \le V_0^2 \sigma_0^2, \\ & w_0 + \mathbf{w}^{\mathrm{T}} \mathbf{1} \le V_0, \\ & w_0 \ge 0, w_1 \ge 0, w_2 \ge 0. \end{array}$$

Here $V_0 = 10^6$ is the initial capital and $V_0 \sigma_0 = 25000$.
If we, for now, ignore the short-selling constraints, then the sufficient conditions for optimality are

(1) $R_0 - \lambda_2 = 0$ and $-\boldsymbol{\mu} + \lambda_1 \Sigma \mathbf{w} + \lambda_2 \mathbf{1} = 0$,

(2) $\mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w} \le V_0^2 \sigma_0^2$ and $w_0 + \mathbf{w}^{\mathrm{T}} \mathbf{1} \le V_0$

(3) $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,

(4) $\lambda_1(\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} - V_0^2\sigma_0^2) = 0$ and $\lambda_2(w_0 + \mathbf{w}^{\mathrm{T}}\mathbf{1} - V_0) = 0$.

Assuming $\lambda_1 > 0$ and $\lambda_2 > 0$ leads to $\lambda_2 = R_0$,

$$\mathbf{w} = \frac{1}{\lambda_1}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1}),$$

by (1) and using the first condition in (4) gives

$$V_0^2\sigma_0^2 = \frac{1}{\lambda_1^2}(\boldsymbol{\mu} - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1}).$$

Then we solve for λ_1 which gives

$$\lambda_1 = \frac{1}{V_0\sigma_0}\Big((\boldsymbol{\mu} - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})\Big)^{1/2}$$

and

$$\begin{aligned}\mathbf{w} &= \frac{V_0\sigma_0}{\Big((\boldsymbol{\mu} - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})\Big)^{1/2}}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1}),\\ w_0 &= V_0 - w_1 - w_2.\end{aligned}$$

We can compute

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{pmatrix},$$

and putting in the numerical values gives

$$w_1 = w_2 = 107253.1, \qquad w_0 = 785493.8.$$

Since the solution to the optimization problem without short-selling constraints actually satisfies the short-selling constraints we conclude that this is the optimal solution to the problem.

(b) The mean and standard deviation of the optimal portfolio are given by

$$\begin{aligned} w_0R_0 + (w_1 + w_2)\mu &= 1050231,\\ \sqrt{\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}} &= 25000.\end{aligned}$$

Problem 5

Let $X = V_1 - V_0R_0$ be the net worth. Then the discounted loss is

$$\begin{aligned} L &= -X/R_0\\ &= -\frac{1}{R_0}\Big(800000R_0 + 100000(R_1 + R_2) - 1000000R_0\Big)\\ &= 200000 - \frac{100000}{R_0}(R_1 + R_2)\\ &= 200000 - \frac{100000R_0}{R_0(1-q)}((1 - I_1) + (1 - I_2))\\ &= -200000\frac{q}{1-q} + \frac{100000}{1-q}(I_1 + I_2).\end{aligned}$$

The distribution of $I_1 + I_2$ is given by

$$\begin{aligned}
P(I_1 + I_2 = 0) &= (1-p)^2, \\
P(I_1 + I_2 = 1) &= 2p(1-p), \\
P(I_1 + I_2 = 2) &= p^2,
\end{aligned}$$

and the quantile function is therefore given by

$$F^{-1}_{I_1+I_2}(1-u) = \begin{cases} 0, & \text{if } 1-u \le (1-p)^2, \\ 1, & \text{if } (1-p)^2 < 1-u \le 1-p^2, \\ 2, & \text{if } 1-p^2 < 1-u. \end{cases}$$

The Value-at-Risk is then given by

$$\begin{aligned}
\mathrm{VaR}_u(X) &= F_L^{-1}(1-u) \\
&= -200000\frac{q}{1-q} + \frac{100000}{1-q} F^{-1}_{I_1+I_2}(1-u) \\
&= -200000\frac{q}{1-q} + \frac{100000}{1-q} \begin{cases} 0, & \text{if } 1-(1-p)^2 \le u, \\ 1, & \text{if } p^2 \le u < 1-(1-p)^2, \\ 2, & \text{if } u < p^2. \end{cases}
\end{aligned}$$

(a) With $u = 0.05$ and $p = 0.024$ we have $1-(1-p)^2 = 0.04742 < 0.05$ and $p^2 = 0.000576$ and therefore

$$\mathrm{VaR}_{0.05}(X) = -200000q/(1-q) = -5128.2.$$

(b) The Expected Shortfall can be computed as

$$\begin{aligned}
\mathrm{ES}_{0.05}(X) &= \frac{1}{0.05}\int_0^{0.05} \mathrm{VaR}_u(X)du \\
&= -200000\frac{q}{1-q} + \frac{100000}{0.05(1-q)}\Big(2(p^2-0) + 1(1-(1-p)^2-p^2)\Big) \\
&= 93333.
\end{aligned}$$

(c) Let $I_1 = 1$ with probability $p_1 = 0.91$ and I_2 unchanged. Then

$$\begin{aligned}
P(I_1 + I_2 = 0) &= (1-p_1)(1-p) = 0.08784, \\
P(I_1 + I_2 = 1) &= p_1(1-p) + (1-p_1)p = 0.89032, \\
P(I_1 + I_2 = 2) &= p_1 p = 0.02184,
\end{aligned}$$

and

$$F^{-1}_{I_1+I_2}(1-u) = \begin{cases} 0, & \text{if } 1-u \le (1-p_1)(1-p), \\ 1, & \text{if } (1-p_1)(1-p) < 1-u \le 1-p_1p, \\ 2, & \text{if } 1-p_1p < 1-u. \end{cases}$$

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&= -200000\frac{q}{1-q} + \frac{100000}{1-q} \begin{cases} 0, & \text{if } 1-(1-p_1)(1-p) \le u, \\ 1, & \text{if } p_1p \le u < (1-p_1)(1-p), \\ 2, & \text{if } u < p_1p. \end{cases}
\end{aligned}$$

In particular

$$\mathrm{VaR}_{0.05}(X) = -200000q/(1-q) + \frac{100000}{1-q} = 97436.$$

The Expected Shortfall is given by

$$\begin{aligned}
\mathrm{ES}_{0.05}(X) &= \frac{1}{0.05}\int_0^{0.05} \mathrm{VaR}_u(X)du \\
&= -200000\frac{q}{1-q} + \frac{100000}{0.05(1-q)}\Big(2(p_1p - 0) + 1(0.05 - p_1p)\Big) \\
&= 142236
\end{aligned}$$