KTH Matematik

EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT, 2012-02-11.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

Consider a portfolio with random value V_1 at time 1. The certainty equivalent is the fixed amount C at time 1 for which the investor, at time 0, is indifferent between receiving the fixed amount C or the random amount V_1 at time 1. If the investor maximizes expected utility, with utility function u, then the certainty equivalent is the number C such that $u(C) = E[u(V_1)]$. If the utility function is strictly increasing, then the inverse u^{-1} exists and $C = u^{-1}(E[u(V_1)])$.
A simple example is when V_1 can take only two values; $P(V_1 = 0) = 1/2$ and $P(V_1 = 1) = 1/2$. If the utility function is $u(x) = \log(1 + x)$, then the expected utility is

$$E[\log(1 + V_1)] = 0.5\Big(\log(1 + 0) + \log(1 + 1)\Big) = 0.5 \log 2.$$

Since $u^{-1}(y) = e^y - 1$, the certainty equivalent is

$$C = e^{0.5 \log 2} - 1 = \sqrt{2} - 1.$$

Problem 2

Let $0 = t_0 < t_1 < \cdots < t_n$ be times and let the corresponding discount factors be $d_1, \dots, d_n$.
An interest rate swap is an agreement between two parties to exchange floating interest rate payments for a fixed interest rate payment. The floating interest rate payment between the times t_{k-1} and t_k paid at t_k is

$$L\Big(\frac{1}{d_{k-1,k}} - 1\Big), \tag{1}$$

where L is the principal and $d_{k-1,k}$ is the discount factor at time t_{k-1} between times t_{k-1} and t_k. The fixed receiver receives the fixed amount cL at each of the times

$t_1, \ldots, t_n$ and in return pays the floating interest rate payment at each of those times. The fixed coupon c is determined so that the initial value of the swap is zero.

A life insurer often have liabilities far into the future, at least 30 years, and is therefore exposed to changes, e.g. parallel shifts, of the zero rate curve. If there is a parallel shift down, the value of the long liabilities goes up. To become immune to parallel shifts the life insurer would need to buy bonds with maturity 30 years or more into the future. But such bonds are often not available. An alternative is to enter into swaps as the fixed receiver. By entering a swap as the fixed receiver the value of the swap goes up as the zero rates go down. Therefore, by entering a swap at an appropriate principal the life insurer can become immune to parallel shifts.

Problem 3

The best odds you can get are 4.70 on Everton, 3.70 on Draw, and 1.95 on Manchester City. Betting 213 on Everton at bookmaker 7, 271 on Draw at bookmaker 4 and 513 on Manchester City at bookmaker 3 is an arbitrage. Indeed, then you have paid $213 + 271 + 513 = 997$ and will receive

$$\begin{aligned} 213 \cdot 4.70 &= 1001.1 \quad \text{(Everton)} \\ 271 \cdot 3.70 &= 1002.7 \quad \text{(Draw)} \\ 513 \cdot 1.95 &= 1000.35 \quad \text{(Manchester City)}. \end{aligned}$$

Problem 4

Write $r_{0.5,0.75} = 0.06 + 0.015W$ where W has standard Normal distribution. The price of the 9m-bond in six months is

$$Z = 100\exp\{-r_{0.5,0.75} \cdot 0.25\} = 100\exp\{-0.06 \cdot 0.25 + 0.015 \cdot 0.25 \cdot W\}.$$

Then, since $E[e^{aW}] = e^{a^2/2}$, we have

$$\begin{aligned} E[Z] &= 100\exp\{-0.06 \cdot 0.25 + (0.015 \cdot 0.25)^2/2\}, \\ V(Z) &= E[Z^2] - (E[Z])^2 \\ &= 100^2\exp\{-2 \cdot 0.06 \cdot 0.25 + (2 \cdot 0.015 \cdot 0.25)^2/2\} \\ &\quad - \Big(100\exp\{-0.06 \cdot 0.25 + (0.015 \cdot 0.25)^2/2\}\Big)^2. \end{aligned}$$

Let R denote the return of the 9-month bond in six months from now. Then

$$\begin{aligned} \mu = E[R] &= \frac{E[Z]}{95.10} = 1.035877, \\ \sigma^2 = V(R) &= \frac{V(Z)}{95.10^2} = 1.508974 \cdot 10^{-5}. \end{aligned}$$

The net value of the portfolio is $V_{0.5} = w_0 + w_1 R - 100$, where w_0 and w_1 are the amounts invested in the cash account and 9m-bond, respectively. Thus, the objective is to minimize $w_1^2\sigma^2$ subject to $w_0 + w_1 \leq 97$ and $w_0 + w_1\mu \geq 100$. Clearly, you must take w_1 as small as possible without violating the constraints. This leads to, w_0 and w_1 as solutions to

$$\begin{aligned} w_0 + w_1 &= 97, \\ w_0 + w_1\mu &= 100. \end{aligned}$$

That is, $w_1 = 83.61943$ and $w_0 = 13.38057$, which corresponds to 0.879279 number of 9m-bonds and 13.38057 on the cash account.
(b) The expected value is 0 and the standard deviation is

$$w_1\sigma = 0.3248.$$

Problem 5

(a) $\mathrm{VaR}_{0.01}(V_1 - V_0 R_0) = V_0 + F^{-1}_{-V_1/R_0}(0.99)$, where

$$V_1/R_0 = \begin{cases} -950 & \text{with probability } 0.95, \\ -855 & \text{with probability } 0.045, \\ 0 & \text{with probability } 0.005. \end{cases}$$

Thus, $F^{-1}_{-V_1/R_0}(0.99) = -855$ and $\mathrm{VaR}_{0.01}(V_1 - V_0 R_0) = 940 - 855 = 85$.
(b)

$$\begin{aligned} \mathrm{ES}_{0.01}(V_1 - V_0 R_0) &= \frac{1}{\alpha}\int_0^{0.01} \mathrm{VaR}_u(V_1 - V_0 R_0)du \\ &= \frac{1}{0.01}\Big(85 \cdot (0.995 - 0.99) + 940 \cdot 0.005\Big) = 512.5. \end{aligned}$$