KTH Matematik

EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT, 2012-10-19.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

- Translation invariance: $\rho(X + cR_0) = \rho(X) - c$, $c \in (0, \infty)$. Adding a capital c at time 0 and investing it in the risk-free asset reduces the risk by c.
- Monotonicity: $X_1 \leq X_2$ implies $\rho(X_1) \geq \rho(X_2)$. If X_2 is worth more than X_1 no matter the outcome, then X_2 has smaller risk.
- Convexity: $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$, $\lambda \in [0, 1]$. Take for example, $\rho(X_1) = \rho(X_2)$. Then spreading the risk between the two assets reduces the risk. The risk measure rewards diversification.
- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$, $\lambda \geq 0$. Doubling the position doubles the risk.
- Subadditivity:$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$. Think of X_1 and X_2 as portfolios of two business lines of a company. Then, breaking up the company in two pieces increases the risk.

Problem 2

Let C be the $m \times n$ cash-flow matrix of where each row contains the cash-flow payments of the bonds. m is the number of bonds and n the number of cash-flow times. If p is the price vector, then the vector of discount factors d is a solution to $Cd = p$. The typical situation is that m is smaller than n leading to infinitely many solutions. The bootstrapping procedure is used to find a reasonable solution. The bootstrapping procedure works as follows. Start by obtaining the discount factors for the maturities of the zero coupon bonds. In the example it corresponds to bond A. The cash flow times are $t_1 = 8/12, t_2 = 20/12, t_3 = 32/12, t_4 = 44/12$. Then

$1035573 = 1.0425 \cdot 10^6 d_1$ which gives $d_1 = 0.9934$ and $r_1 = -(1/t_1)\log(d_1) = 0.010$. With d_1 determined we can obtain d_2 from bond B. We get

$$p_B = 0.04 \cdot 10^6 d_1 + 1.04 \cdot 10^6 d_2,$$

which leads to $d_2 = 0.9769$ and $r_2 = 0.014$. Now, only one bond remains, but two unknown discount factors. Then, one assumes d_3 is given by linear interpolation between d_2 and d_4 so

$$d_3 = d_2 + \frac{d_4 - d_2}{t_4 - t_2}(t_3 - t_2).$$

This is inserted into the equation for bond C given by

$$p_C = 0.0325 \cdot 10^6(d_1 + d_2 + d_3) + 1.0325 \cdot 10^6 d_4.$$

Solving for d_4 gives $d_4 = 0.9361$, $r_4 = 0.18$, and then $d_3 = 0.9565$ and $r_3 = 0.0167$.

Problem 3

Because the interest rate is known the futures price F_0 equals the forward price G_0. Indeed, using the "futures strategy" a zero initial capital can generate the payment $e^{r_1+\cdots+r_T}(S_T - F_0)$ at time T. Similarly, a long position in $e^{r_1+\cdots+r_T}$ forward contracts generate $e^{r_1+\cdots+r_T}(S_T - G_0)$. Therefore we must have $G_0 = F_0$.
The price of the call option is then given as $C_0(F_0)$ using Black's formula, with G_0 replaced by F_0. The delta hedge of a call option is then to take a long position of size $\Delta = \frac{\partial}{\partial F_0} C_0(F_0)$ in the futures contract. The Δ is computed as (where $B_0 = e^{-(r_1+\cdots+r_T)}$)

$$\begin{aligned}\Delta_C &= \frac{\partial}{\partial F_0} C_0(F_0) \\ &= B_0\Phi(d_1) + B_0\Big(F_0 \frac{\partial}{\partial F_0}\Phi(d_1) - K\frac{\partial}{\partial F_0}\Phi(d_2)\Big).\end{aligned}$$

The second term vanishes because $\frac{\partial}{\partial F_0} d_1 = \frac{\partial}{\partial F_0} d_2$ and $d_2^2 = d_1^2 - 2\log(F_0/K)$ leads to

$$\begin{aligned}F_0 \frac{\partial}{\partial F_0}\Phi(d_1) - K\frac{\partial}{\partial F_0}\Phi(d_2) &= (F_0\varphi(d_1) - K\varphi(d_2))\frac{\partial}{\partial F_0} d_1 \\ &= (F_0 - Ke^{\log(F_0/K)})\frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}\frac{\partial}{\partial F_0} d_1 = 0.\end{aligned}$$

Here φ denotes the standard normal density. Thus, the delta hedge of a call option is given by $\Delta_C = B_0\Phi(d_1)$.
By the put-call-parity

$$C_0 - P_0 = B_0(F_0 - K),$$

which implies that the delta hedge of a put option is

$$\Delta_P = \frac{\partial}{\partial F_0} P_0(F_0) = \Delta_C - B_0 = B_0(\Phi(d_1) - 1).$$

Since there is no cost in entering the futures contract there premium from the option P_0 is put in to the bank account. As time passes the position in the futures contracts are updated accordingly and the resettlement payments are put into the bank account.

Problem 4

As the given set of digital options have overlapping ranges the "horse race" setting does not apply directly. However, an alternative set of digital options with disjoint ranges can be constructed from the given digitals. The construction is summarized in Table 1. For example, an option with range $1.2-1.5$ is constructed by a long position in option E and a short position in option D. The price of this option is $49.26-34.05=15.21$. The corresponding forward price for a contract paying 1 if the underlying falls in the indicated range is $q_5=15.21/(0.9975\cdot 100)=0.1525$. The subjective probability is $\Phi((0.015-0.01)/0.0025)-\Phi((0.012-0.01)/0.0025)=\Phi(2)-\Phi(0.8)=0.1891$. The other options are constructed similarly. With the options in Table 1 the investment problem is a standard "horse race" problem, which amounts to

New option label	Range	Construction	Price	q_k	p_k
1	$(-\infty,0.5]$	A-B+C-D	0.62	0.0062	0.0228
2	$(0.5,0.7]$	B-C+D	6.02	0.0604	0.0923
3	$(0.7,1.0]$	C-D	43.23	0.4333	0.3849
4	$(1.0,1.2]$	D	34.05	0.3414	0.2881
5	$(1.2,1.5]$	E-D	15.21	0.1525	0.1891
6	$(1.5,\infty)$	F-E+D	0.62	0.0062	0.0228

Table 1: New digital options.

$$\begin{array}{rl}\text{maximize} & E[u(\sum_{k=1}^{n}\frac{w_k}{q_k}X_k)]\\ \text{subject to} & w_1+\cdots+w_n\le V_0/B_0.\end{array}$$

The solution, when $\tau=0$, is to invest the amount w_k in the kth digital, where

$$w_k=V_0\frac{q_k(p_k/q_k)^{\gamma}}{\sum_{j=1}^{n}q_j(p_j/q_j)^{\gamma}}.$$

With the data given in Table 1 the solution is:

$$w_1=33.15, w_2=9.820, w_3=8.015, w_4=5.150, w_5=10.72, w_6=33.15.$$

The corresponding number of each "new" digital in the portfolio is $h_k=w_k/q_k$ Translated to the original options the optimal investment is to buy the following number of original options:

$$\begin{aligned}&h_A=h_1,\ h_B=h_2-h_1,\ h_C=h_3-h_2+h_1,\\ &h_D=-h_1+h_2-h_3+h_4-h_5+h_6,\ h_E=h_5-h_6,\ h_F=h_6.\end{aligned}$$

The corresponding amounts invested in the original options are

$$w_A=355, w_B=-2553, w_C=4020, w_D=30.38, w_E=-2599, w_F=846.4.$$

Problem 5

The price in three months of the bonds can be written as

$$P_3^A = 1.0425 \cdot 10^6 e^{-r_{3,8}(5/12)} = \mathbb{1}.0425 \cdot 10^6 B_5 R_5,$$
$$P_3^B = 0.04 \cdot 10^6 e^{-r_{3,8}(5/12)} + 1.04 \cdot 10^6 e^{-r_{3,20}(17/20)} = 0.04 \cdot 10^6 B_5 R_5 + 1.04 \cdot 10^6 B_{17} R_{17}.$$

Therefore, the corresponding three-month returns can be written as

$$\begin{aligned} R^A &= \frac{P_3^A}{P_0^A} = \frac{1.0425 \cdot 10^6 B_5}{P_0^A} R_5 \\ R^B &= \frac{P_3^B}{P_0^B} = \frac{0.04 \cdot 10^6 B_5}{P_0^B} R_5 + \frac{1.04 \cdot 10^6 B_{17}}{P_0^B} R_{17}. \end{aligned}$$

In vector form this relation is written

$$\begin{pmatrix} R^A \\ R^B \end{pmatrix} = M \begin{pmatrix} R^5 \\ R^{17} \end{pmatrix}, \text{where } M = \begin{pmatrix} \frac{1.0425 \cdot 10^6 B_5}{P_0^A} & 0 \\ \frac{0.04 \cdot 10^6 B_5}{P_0^B} & \frac{1.04 \cdot 10^6 B_{17}}{P_0^B} \end{pmatrix}.$$

Therefore the covariance matrix of $(R^A, R^B)^T$ is equal to

$$\Sigma_{AB} = M \Sigma M^T.$$

The objective can be written as

$$\begin{aligned} &\text{minimize } \frac{1}{2} w^T \Sigma_{AB} w, \\ &\text{subject to } w^T 1 \leq -V_0. \end{aligned}$$

where $w = (w_A, w_B)^T$ and $V_0 = 1.75 \cdot 10^9$. The solution is found by solving the two equations:

$$\begin{aligned} \Sigma_{AB} w + \lambda 1 &= 0, \\ 1^T w &\leq -V_0, \end{aligned}$$

which yields

$$w = -\frac{\Sigma_{AB}^{-1} 1}{1^T \Sigma_{AB}^{-1} 1}.$$

Numerically we compute (rounded to two decimals)

$$\Sigma_{AB} = 10^{-5} \begin{pmatrix} 2.60 & 2.57 \\ 2.57 & 3.08 \end{pmatrix}, \ \Sigma_{AB}^{-1} = 10^5 \begin{pmatrix} 2.18 & -1.82 \\ -1.82 & 1.85 \end{pmatrix}, w = -10^9 \begin{pmatrix} 1.64 \\ 0.11 \end{pmatrix}.$$