KTH Matematik

EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT, 2013-01-09.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

Let f_k, $k = 1, \dots, n$ be the payoff functions of the different call and put options available on the market. Portfolios with payoff $f(S_T) = \sum_{k=1}^n h_k f_k(S_T)$ can be formed on the market. A portfolio f is an arbitrage if the price of f is 0, $P(f(S_T) \geq 0) = 1$, and $P(f(S_T) > 0) > 0$.
(a) The *no-arbitrage theorem* states that if there are no arbitrage opportunities, then there exists an implied forward distribution such that forward prices of the options can be expressed as expectations under the implied forward distribution.
(b) Suppose we have call prices at different strikes: $C_1 = C(K_1), \dots, C_n = C(K_n)$. One way to obtain an implied forward distribution is to determine the implied volatilities $\sigma(K_1), \dots, \sigma(K_n)$ by solving

$$C_j = C^B(K_j, \sigma(K_j)),$$

where $C^B(K, \sigma)$ is Black's formula for call options as a function of the strike K and volatility σ. An implied volatility curve $\sigma(K)$ can be fitted to the $\sigma(K_j)$, $j = 1, \dots, n$ which then results in the call price function $C(K) = B_0 C^B(K, \sigma(K))$. Since prices are discounted expectations under the forward distribution Q it follows that

$$C(K) = B_0 E_Q[(S_T - K)_+] = B_0 \int_K^\infty (x - K) q(x) dx,$$

where q is the density of the forward distribution. Taking derivatives with respect to K it follows that

$$\frac{\partial C}{\partial K}(K) = -B_0(1 - Q(K)),$$

from which the implied forward distribution can be obtained as

$$Q(K) = \frac{1}{B_0} \frac{\partial C}{\partial K}(K) + 1.$$

Problem 2

(a) The future net worth is $X = V_1 - V_0 R_0$ and the Value-at-Risk at level $p \in (0,1)$ is defined as

$$\mathrm{VaR}_p(X) = \min\{m : P(X + mR_0 < 0) \leq p\} = \min\{m : P(V_T + (m - V_0)R_0 < 0) \leq p\}.$$

Writing $L = -X/R_0 = V_0 - V_T/R_0$ the Value-at-Risk can alternatively be defined as

$$\mathrm{VaR}_p(X) = \min\{m : P(L \leq m) \geq 1 - p\} = F_L^{-1}(1 - p).$$

(any of these expression or equivalent expression give full credit)
(b) The Expected Shortfall at level $p \in (0,1)$ is defined as

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)du.$$

(c) With the second representation of Value-at-Risk in (a) the Expected Shortfall can be expressed as

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)du = \frac{1}{p}\int_0^p F_L^{-1}(1-u)du.$$

(d) Take $\lambda > 0$. Expected Shortfall is positively homogeneous because $F_{\lambda L}^{-1}(1-u) = \lambda F_L^{-1}(1-u)$ for each u and therefore

$$\mathrm{ES}_p(\lambda X) = \frac{1}{p}\int_0^p F_{\lambda L}^{-1}(1-u)du = \lambda\frac{1}{p}\int_0^p F_L^{-1}(1-u)du = \lambda\,\mathrm{ES}_p(X).$$

Problem 3

Let V denote the value of the car, $f = 0.5$ the fraction lost in case of an accident, $c = 0.02$ the premium as a fraction of the car's value, and $p = 0.035$ the probability of an accident. In case of insurance the value at the end of the year is

$$V_1 = V(1 - c),$$

whereas in case of no insurance the value at the end of the year is

$$V_1 = V(1 - I) + V(1 - f)I,$$

where I is the accident indicator. The expected utility in each case is

$$\begin{aligned} &V^\beta(1-c)^\beta, \text{ with insurance,} \\ &V^\beta[(1-p) + (1-f)^\beta p], \text{ without insurance.} \end{aligned}$$

The fact that the car owner does not buy the insurance implies that

$$g(\beta) = (1-p) + (1-f)^\beta p - (1-c)^\beta > 0.$$

It remains to determine for which values of β that $g(\beta) > 0$. Let us write

$$g(\beta) = 1 - p + pe^{\beta \log(1-f)} - e^{\beta \log(1-c)}$$

and

$$g'(\beta) = p \log(1-f) e^{\beta \log(1-f)} - \log(1-c) e^{\beta \log(1-c)}$$

The sign of $g'(\beta)$ depends on which term that dominates. Note that, with the given numerical values $g(0) = 0$, $g'(0) < 0$ but eventually $g'(\beta) > 0$ (when the first term in the expression of g' dominates) and then g is increasing from then on. We conclude that there is at most one point $\beta^* \in (0, 1]$ where $g(\beta) > 0$ for $\beta > \beta^*$. Numerically we can find $\beta^* \approx 0.56$.

Problem 4

(a) Write $\boldsymbol{\mu} = \mu_0 \mathbf{1}$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & \dots \\ 0 & \ddots & \\ \vdots & & \sigma_n^2 \end{pmatrix}, \qquad \Sigma^{-1} = \begin{pmatrix} \sigma_1^{-2} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & \sigma_n^{-2} \end{pmatrix}.$$

The solution to the trade-off problem is (with the standard optimization procedure, see Proposition 4.1 p. 88 for details)

$$\mathbf{w} = \frac{V_0}{c} \Sigma^{-1} \Big(\boldsymbol{\mu} - \frac{(\mathbf{1}^{\mathrm{T}} \Sigma^{-1} \boldsymbol{\mu} - c)_+}{\mathbf{1}^{\mathrm{T}} \Sigma^{-1} \mathbf{1}} \mathbf{1} \Big).$$

If $c \geq \mathbf{1}^{\mathrm{T}} \Sigma^{-1} \boldsymbol{\mu} = \mu_0 \sum_{k=1}^n \sigma_k^{-2}$, then the solution is

$$\mathbf{w} = \frac{V_0}{c} \Sigma^{-1} \boldsymbol{\mu} = \frac{V_0 \mu_0}{c} \begin{pmatrix} \sigma_1^{-2} \\ \vdots \\ \sigma_n^{-2} \end{pmatrix},$$

which is a solution where the investor is too risk averse and "throws away money". If $c < \mathbf{1}^{\mathrm{T}} \Sigma^{-1} \boldsymbol{\mu} = \mu_0 \sum_{k=1}^n \sigma_k^{-2}$, then the solution is

$$\begin{aligned} \mathbf{w} &= \frac{V_0}{c} \Sigma^{-1} \Big(\boldsymbol{\mu} - \frac{(\mathbf{1}^{\mathrm{T}} \Sigma^{-1} \boldsymbol{\mu} - c)}{\mathbf{1}^{\mathrm{T}} \Sigma^{-1} \mathbf{1}} \mathbf{1} \Big) \\ &= V_0 \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^{\mathrm{T}} \Sigma^{-1} \mathbf{1}}. \end{aligned}$$

That is,

$$w_k = V_0 \frac{1/\sigma_k^2}{\sum_{j=1}^n 1/\sigma_j^2},$$

so the investor invests proportional to the reciprocal of the variance.
(b) If $c \geq \mathbf{1}^{\mathrm{T}} \Sigma^{-1} \boldsymbol{\mu}$, then the mean is

$$\mathbf{w}^{\mathrm{T}} \boldsymbol{\mu} = \frac{V_0 \mu_0^2}{c} \sum_{j=1}^n \sigma_j^{-2}$$

and the variance is

$$\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} = \Big(\frac{V_0\mu_0}{c}\Big)^2 \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\Sigma\Sigma^{-1}\mathbf{1} = \Big(\frac{V_0\mu_0}{c}\Big)^2 \sum_{j=1}^{n}\sigma_j^2.$$

If $c < \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{\mu}$, then the mean is

$$\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} = V_0\mu_0,$$

and the variance is

$$\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} = V_0^2\frac{\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\Sigma\Sigma^{-1}}{(\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1})^2} = V_0^2\frac{1}{\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1}} = \frac{V_0^2}{\sum_{j=1}^{n}\sigma_j^{-2}}.$$

(c) The efficient frontier is the curve $\{(\sigma(c), \mu(c)) : c > 0\}$. For $c < \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{\mu}$ it is just a single point since $\sigma(c)$ and $\mu(c)$ does not depend on c in this case. For $c \geq \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{\mu}$ we have

$$\mu(c) = \frac{V_0\mu_0^2}{c}\sum_{j=1}^{n}\sigma_j^{-2} = \sigma(c)\mu_0\frac{\sum_{j=1}^{n}\sigma_j^{-2}}{\Big(\sum_{j=1}^{n}\sigma_j^2\Big)^{1/2}},$$

so the efficient frontier is linear.

Problem 5

The objective is to hedge the liability $L = B(7,12)$ with the asset $B(7,18)$ and a cash position (that does not pay interest). Write the hedging portfolio as $h_0 + hB(7,18)$. The quadratic hedge is then to take

$$h = \frac{\mathrm{Cov}(B(7,18), B(7,12))}{\mathrm{Var}(B(7,18))}, \quad h_0 = E[B(7,12)] - hE[B(7,18)].$$

It remains to compute the relevant covariance, variance, and expectations.
We can write $Z_1 + \cdots + Z_7 \stackrel{\mathrm{d}}{=} \sqrt{7}Z$ where $Z \sim N(0,1)$. Since, for any real numbers a and b $E[e^{aZ}] = e^{a^2/2}$ it follows that

$$\mathrm{Cov}(e^{aZ}, e^{bZ}) = E[e^{(a+b)Z}] - E[e^{aZ}]E[e^{bZ}] = e^{(a+b)^2/2} - e^{a^2/2+b^2/2} = e^{a^2/2+b^2/2}(e^{ab} - 1),$$

and

$$\mathrm{Var}(e^{aZ}) = \mathrm{Cov}(e^{aZ}, e^{aZ}) = e^{a^2}(e^{a^2} - 1).$$

With these formulas we compute

$$\begin{aligned}\mathrm{Cov}(B(7,18), B(7,12)) &= 5.38 \cdot 10^{-6},\\ \mathrm{Var}(B(7,18)) &= 1.17 \cdot 10^{-5},\\ E[B(7,12)] &= 0.9906,\\ E[B(7,18)] &= 0.9800.\end{aligned}$$

The resulting quadratic hedge is

$$h_0 = 0.54, \quad h = 0.46.$$

(b) The cost of the quadratic hedge is

$$h_0 + hB(0,18) = 0.9862$$

which is higher than the current value of the liability $B(0,12) = 0.9810$.