SF2943 Time Series Analysis
Problem 6: Application of the GARCH model

Before you start solving this problem, download and open the workspace and read the workspace guide. Both are available from Alexander Aurell's homepage (link on the course website).

Background

Your task is to model (with the help of Quantlab) the risk of holding one unit's worth of the OMXS30 index for a few days using a GARCH process and compare it to a naive approach. The OMXS30 index is a weighted mean of the 30 most traded stocks on the Stockholm stock exchange. Let S_t be the value of the index at closing time day t and let $t = 0$ be today. A typical thing to look at when evaluating risk of holding a unit of the index for $k > 0$ days is the quantiles in the left tail of $S_k - S_0$. This gives an estimate of the worst case return (loss of money) in a certain percentage of all possible scenarios.

The distribution of the return on the investment should be modeled in two ways: A naive approach based on fitting a normal distribution and using a GARCH process.

First, transform past index values $(S_{-N}, \ldots, S_0)$ into log-returns,

$$X_t = \ln(S_t/S_{t-1}). \tag{1}$$

For the naive approach, assume that the log-returns are i.i.d. $\mathcal{N}(\mu, \sigma)$-distributed random variables. Estimate (μ, σ) using historical data and use the fact that

$$S_k - S_0 = S_0 \left(e^{X_1 + \cdots + X_k} - 1\right) \overset{d}{\approx} S_0 \left(e^{k\hat{\mu} + \sqrt{k}\hat{\sigma} Z} - 1\right), \tag{2}$$

where $(\hat{\mu}, \hat{\sigma})$ are the obtained estimates of the parameters in the normal distribution and Z is a standard normal random variable. By sampling from Z, the empirical quantiles of $-(S_k - S_0) = S_0 - S_k$ can be calculated approximately. Furthermore, there are exact formulae for the quantile,

$$F^{-1}_{S_0 - S_k}(p) = S_0 \left(1 - e^{k\hat{\mu} - \sqrt{k}\hat{\sigma}\Phi^{-1}(p)}\right), \quad p \in (0, 1), \tag{3}$$

and for the density of $S_k - S_0$,

$$f_{S_k - S_0}(x) = \left| \frac{1}{\sqrt{2\pi} S_0 \sqrt{k} \hat{\sigma} (1 + x/S_0)} \right| \exp\left(- \frac{\left(\ln(1 + x/S_0) - k\hat{\mu}\right)^2}{2k\hat{\sigma}^2} \right). \tag{4}$$

For the second approach, a GARCH process should be fitted to the log-returns of $(S_{-N}, \ldots, S_0)$ and S_k is retrieved through the following scheme:

1. $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \beta_j X_{t-j}^2$,

2. $X_t = \sigma_t Z_t, \quad Z_t \sim \mathcal{N}(0,1)$,

3. $Y_t = \mu + X_t$ (here you may use $\mu = 0$ when solving the exercises),

4. $S_k = S_0 \exp(Y_1 + \cdots + Y_k)$.

From simulations it is now possible to calculate the empirical quantile of

$$-(S_k - S_0) = S_0 \left(1 - \exp(Y_1 + \cdots + Y_k)\right). \tag{5}$$

Problems

a) Suppose that you yesterday invested one units worth of money in the OMXS30 index and are planning on selling today. Calculate the 0.05-quantiles this investment with the two approaches described in the previous section. Your answer should be given in Swedish Krona.

b) Calculate or simulate the 0.05-quantiles for a 10-day investment in one units worth of the OMXS30 index, using 250 business days of historical data, if "today" is

i) 2011-08-18,

ii) 2014-04-24,

iii) 2015-06-18.

Your answers should be given in Swedish Krona.

c) In the Quantlab workspace you will find a plot of the log-returns together with the GARCH and the naive volatility of a 1-day investment. Use this plot to interpret your results from b).