SF2943 TIME SERIES ANALYSIS: A FEW COMMENTS ON SPECTRAL DENSITIES

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The aim of this brief note is to get some basic understanding of spectral densities by computing, plotting and interpreting the spectral densities of the time series

$$X_t = 0.8X_{t-1} + Z_t + 0.9Z_{t-1}, \tag{1}$$

$$X_t = -0.8X_{t-1} - 0.9X_{t-2} + Z_t, \tag{2}$$

$$X_t = -0.8X_{t-1} + Z_t, \tag{3}$$

$$X_t = 0.8X_{t-1} + Z_t, \tag{4}$$

where $\{Z_t\}$ is an iid sequence of standard normal random variables. The processes are examples of causal linear time series, the roots of the autoregressive polynomials are all located outside the unit circle in the complex plane.

In order to get some very preliminary feeling for the four systems above, we study how the linear systems transforms the input $(z_1, z_2, z_3, \dots) = (1, 0, 0, \dots)$ into output values $(x_1, x_2, x_3, \dots)$. The result (with linear interpolation) is shown in Figure 1. The AR(1) process (3) shows a periodic behavior, although a rather trivial one (plus, minus, plus, minus, etc.). The AR(2) process (2) shows a more interesting periodic behavior, with cycle lengths of approximately 3 time steps.

The spectral densities

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h),$$

for $\lambda \in [0, \pi]$, are shown in Figure 2. The AR(1) process (3) has a spectral density with a maximum at $\lambda = \pi$. This is consistent with the cycle length of $2\pi/\pi = 2$ shown in Figure 1. The AR(2) process (2) has a spectral density with a maximum at $\lambda \approx 2$. This is consistent with the cycle length of $2\pi/2 = \pi \approx 3$ shown in Figure 1. The processes (1) and (4) have spectral densities with maxima at $\lambda = 0$. They show no periodic behavior (or cycles of infinite length). Since

$$\lambda(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda,$$

we notice that $\gamma(0) = \int_{-\pi}^{\pi} f(\lambda) d\lambda$. In particular, the comparison of the spectral densities might be visualized more clearly by dividing the spectral densities by the stationary variance $\gamma(0)$ of the respective processes. The relevant expressions for $\gamma(0)$ on pages 89 and

91 in the course text book can be used. Alternatively, we may simulate the processes and estimate the stationary variances by their sample analogues.

Finally, we check that the periodograms based on simulated data from the four time series models produce estimates of the spectral densities that are sufficiently accurate to draw the same conclusions as if we knew the spectral densities. The result is shown in Figure 4.

R-code

```
ar2specdens<-function(l,s,phi1,phi2)
{
s^2/(2*pi*(1+phi1^2+phi2^2+2*phi2+2*(phi1*phi2-phi1)*cos(l)-4*phi2*cos(l)^2))
}

arma11specdens<-function(l,s,phi,theta)
{
s^2*(1+theta^2+2*theta*cos(l))/(2*pi*(1+phi^2-2*phi*cos(l)))
}

xvals<-(0:100)*pi/100
plot(xvals,arma11specdens(xvals,1,0.8,0.9),type="l",xlab="",ylab="",
ylim=c(0,max(ar2specdens(xvals,1,-0.8,-0.9))))
lines(xvals,ar2specdens(xvals,1,-0.8,-0.9))
lines(xvals,ar2specdens(xvals,1,-0.8,0))
lines(xvals,ar2specdens(xvals,1,0.8,0))

ar2roots<-polyroot(c(1,0.8,0.9))

(ar2roots[1]*ar2roots[2])^2/((ar2roots[1]*ar2roots[2]-1)*(ar2roots[2]-ar2roots[1]))*
(ar2roots[1]/(ar2roots[1]^2-1)-ar2roots[2]/(ar2roots[2]^2-1))

g01<-(1+1.7^2/(1-0.8^2))
g02<-(ar2roots[1]*ar2roots[2])^2/((ar2roots[1]*ar2roots[2]-1)*(ar2roots[2]-ar2roots[1]))*
(ar2roots[1]/(ar2roots[1]^2-1)-ar2roots[2]/(ar2roots[2]^2-1))
g03<-1/(1-0.8^2)
g04<-1/(1-0.8^2)

plot(xvals,arma11specdens(xvals,1,0.8,0.9)/g01,type="l",xlab="",ylab="",
ylim=c(0,max(ar2specdens(xvals,1,-0.8,-0.9)/Re(g02))))
lines(xvals,ar2specdens(xvals,1,-0.8,-0.9)/Re(g02))
lines(xvals,ar2specdens(xvals,1,-0.8,0)/g03)
lines(xvals,ar2specdens(xvals,1,0.8,0)/g04)

z<-1

tmpmat<-matrix(0,4,20)
tmpmat[1,1]<-z
tmpmat[2,1]<-z
tmpmat[3,1]<-z
tmpmat[4,1]<-z
```

```
tmpmat[1,2]<-0.8*tmpmat[1,1]+0.9*z
tmpmat[2,2]<--0.8*tmpmat[2,1]
tmpmat[3,2]<--0.8*tmpmat[3,1]
tmpmat[4,2]<-0.8*tmpmat[4,1]

for(i in (3:20))
{
tmpmat[1,i]<-0.8*tmpmat[1,i-1]
tmpmat[2,i]<--0.8*tmpmat[2,i-1]-0.9*tmpmat[2,i-2]
tmpmat[3,i]<--0.8*tmpmat[3,i-1]
tmpmat[4,i]<-0.8*tmpmat[4,i-1]
}

plot(tmpmat[1,],type="l",ylim=c(-1,1.8),xlab="",ylab="")
lines(tmpmat[2,])
lines(tmpmat[3,])
lines(tmpmat[4,])

arma1.sim<-arima.sim(model=list(ar=c(-0.8,-0.9)),n=200)
arma2.sim<-arima.sim(model=list(ar=c(-0.8)),n=200)
arma3.sim<-arima.sim(model=list(ar=c(0.8)),n=200)
arma4.sim<-arima.sim(model=list(ar=c(0.8),ma=c(0.9)),n=200)

spectrum(arma1.sim)
lines(xvals/(2*pi),ar2specdens(xvals,1,-0.8,-0.9))
spectrum(arma2.sim)
lines(xvals/(2*pi),ar2specdens(xvals,1,-0.8,0))
spectrum(arma3.sim)
lines(xvals/(2*pi),ar2specdens(xvals,1,0.8,0))
spectrum(arma4.sim)
lines(xvals/(2*pi),arma11specdens(xvals,1,0.8,0.9))
```

Figures

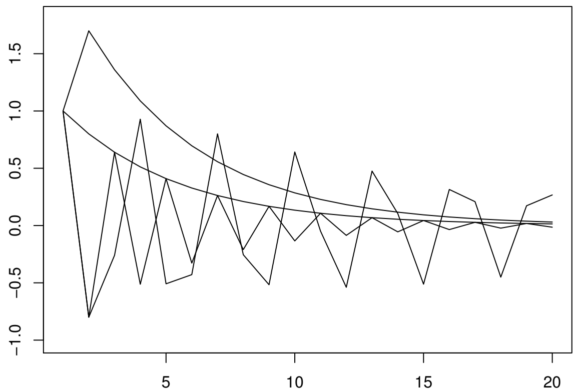

Figure 1. Illustration of how the systems transform input to output

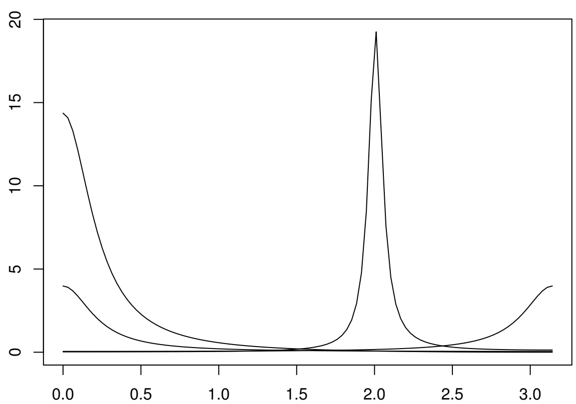

Figure 2. Spectral densities

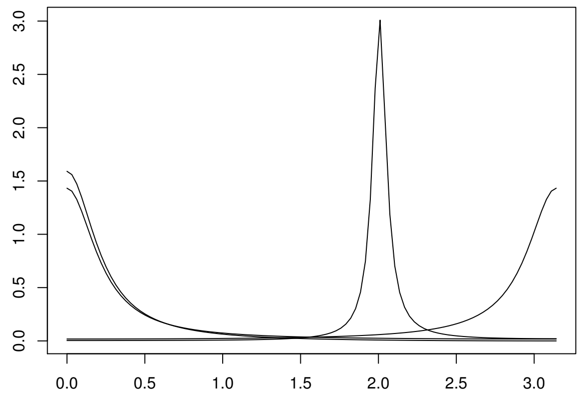

FIGURE 3. Spectral densities normalized by dividing by the stationary variance of the time series

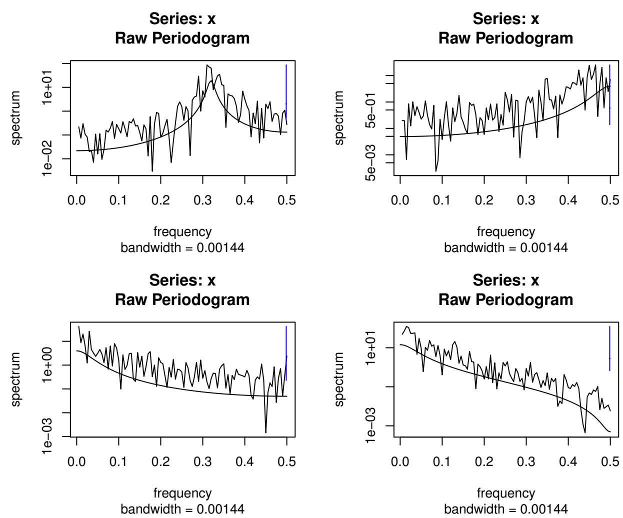

FIGURE 4. Periodograms based on samples of size 200 and the corresponding spectral densities