Notes on the delta method.

Exercise 1, problem 7.

a) this inequality is known as Markov's inequality

b) Show that:

$$\mathbb{P}(|\tau_N - \tau| \geq \varepsilon) = \mathbb{P}((\tau_N - \tau)^2 \geq \varepsilon^2)$$

$$\overset{a)}{\leq} \frac{1}{\varepsilon^2} \mathbb{E}[(\tau_N - \tau)^2] = \frac{1}{\varepsilon^2} \frac{\sigma^2(\phi)}{N} \to 0 \quad \text{as } N \to \infty$$

c) This result is the continuous mapping theorem

the proof relies on introduction of the set B_δ

where $B_\delta = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} : |x - y| < \delta, |\varphi(x) - \varphi(y)| \geq \varepsilon \}$

since φ continuous for every fix ε we have that

$\lim_{\delta \to 0} B_\delta = \emptyset$

Now if $|g(\tau_N) - g(\tau)| > \varepsilon$ then either $|\tau_N - \tau| \geq \delta$ or

$\tau \in B_\delta$

so

$$\mathbb{P}(|g(\tau_N) - g(\tau)| \geq \varepsilon) \leq \mathbb{P}(|\tau_N - \tau| \geq \delta) + \mathbb{P}(\tau \in B_\delta)$$

take the limits, first $N \to \infty$ and then $\delta \to 0$.

d) Use the Taylor expansion around τ

$$\varphi(\tau_N) = \varphi(\tau) + \varphi'(\tau)(\tau_N - \tau) + \frac{\varphi''}{2}(\xi_N)(\tau_N - \tau)^2$$

on the line between τ_N & τ, i.e. $\xi_N \xrightarrow{P} \tau$ as $N \to \infty$

look at

$$\sqrt{N}\left(\varphi(\tau_N) - \varphi(\tau)\right) = \sqrt{N}\,\varphi'(\tau)(\tau_N - \tau) + \frac{\varphi''(\xi_N)}{2}(\tau_N - \tau)\sqrt{N}(\tau_N - \tau)$$

Use previous results together with Slutsky to prove the result.