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**Sf 2955: Computer intensive methods :**
SCALE PARAMETER/ Timo Koski

The notation

$$F(x;\theta)$$

denotes a distribution function that depends on a parameter $\theta$. For example,

$$F(x;\theta) = \begin{cases} 1 - e^{-x/\theta} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

is the distribution function of the exponential distribution $\text{Exp}\,(\theta)$.

# 1 Definition of a Scale Parameter

We define a scale parameter.

**Definition 1.1** Assume $\theta > 0$ in $F(x;\theta)$. Then $\theta$ is a scale parameter, if it holds for all $x$ that

$$F(x;\theta) = H\left(\frac{x}{\theta}\right), \tag{1.1}$$

where $H(x)$ is a distribution function.

■

To take an example, $\theta$ is scale parameter in the exponential distribution $\text{Exp}\,(\theta)$, as we have

$$F(x;\theta) = H\left(\frac{x}{\theta}\right),$$

where

$$H(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

## 2 Properties, Pivotal Variables

We observe two lemmas.

**Lemma 2.1** Assume $X \in F(x;\theta)$. $\theta$ is a scale parameter if and only if the distribution of

$$\frac{X}{\theta}$$

does not depend on $\theta$.

*Proof:* $\Rightarrow$: Assume $\theta$ is a scale parameter. Then the distribution of $\frac{X}{\theta}$ is given by

$$P\left(\frac{X}{\theta} \leq z\right) = P\left(X \leq \theta z\right)$$

since $\theta > 0$ by definition. Then, as we assume that $\theta$ is a scale parameter,

$$P\left(X \leq \theta z\right) = F\left(\theta z;\theta\right) = H\left(\frac{\theta z}{\theta}\right) = H\left(z\right),$$

or

$$P\left(\frac{X}{\theta} \leq z\right) = H(z)$$

which says that $\frac{X}{\theta}$ has distribution which does not depend on $\theta$.

$\Leftarrow$: We assume that $\frac{X}{\theta}$ has distribution, say $H$, which does not depend on $\theta > 0$. Then

$$F(x;\theta) = P\left(X \leq x;\theta\right) = P\left(\frac{X}{\theta} \leq \frac{x}{\theta};\theta\right)$$

since $\theta > 0$. But by assumption

$$P\left(\frac{X}{\theta} \leq \frac{x}{\theta};\theta\right) = H\left(\frac{x}{\theta}\right)$$

where the distribution function $H\left(z\right)$ does not depend on $\theta$. But then we have for any $x$ shown that

$$F(x;\theta) = H\left(\frac{x}{\theta}\right)$$

which in view of (1.1) gives the claim as asserted. ∎

This lemma says that $\frac{X}{\theta}$ is a *pivotal variable.*

**Remark 2.1** Note that a pivotal variable need not be a statistic $-$ the variable can depend on parameters of the model, but its distribution must not. If it is a statistic, then it is known as an *ancillary statistic.* Pivotal quantities are used to the construction of test statistics, e.g., Student's t$-$statistic is pivotal for a normal distribution with unknown variance (and mean). Pivotal variables provide in addition a method of constructing confidence intervals, and the use of pivotal quantities improves performance of the bootstrap, as defined later.

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**Lemma 2.2** Assume $F(x;\theta)$ has a density $\frac{d}{dx}F(x;\theta) = f(x;\theta)$ for all $x$. Then $\theta$ is a scale parameter if and only if

$$f(x;\theta) = \frac{1}{\theta} g\left(\frac{x}{\theta}\right), \tag{2.2}$$

where $g$ is a probability density.

*Proof:* $\Rightarrow$: If $F(x;\theta)$ has an integrable derivative and $\theta$ is a scale parameter, then it holds from (1.1 ) that

$$f(x;\theta) = \frac{d}{dx}F(x;\theta) = \frac{d}{dx}H\left(\frac{x}{\theta}\right) = \frac{1}{\theta}g\left(\frac{x}{\theta}\right)$$

where we have set $g(x) = \frac{d}{dx}H\left(x\right)$, which is a density function, as $H$ is a distribution function.

$\Leftarrow$: We assume that

$$f(x;\theta) = \frac{1}{\theta} g\left(\frac{x}{\theta}\right).$$

Then

$$F(x;\theta) = \int_{-\infty}^{x} f(u;\theta)du = \frac{1}{\theta}\int_{-\infty}^{x} g\left(\frac{u}{\theta}\right)du.$$

Here we make a change of variable $t = \frac{u}{\theta}$, $du = \theta dt$ and thus

$$\frac{1}{\theta}\int_{-\infty}^{x} g\left(\frac{u}{\theta}\right)du = \int_{-\infty}^{\frac{x}{\theta}} g\left(t\right)dt = G\left(\frac{x}{\theta}\right).$$

where $\frac{d}{dx}G\left(x\right) = g(x)$. In other words we have obtained for every $x$ that

$$F(x;\theta) = G\left(\frac{x}{\theta}\right),$$

and since $G(x)$ is a distribution function, the desired assertion follows by (1.1). ■

## 3 Examples and the Scale-Free property

We give first some further examples.

1. $X \in N(\mu\sigma, \sigma^2)$. We guess that $\sigma$ is a scale parameter. The pertinent density is
$$f(x;\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu\sigma)^2}{2\sigma^2}}$$
and some elementary algebra gives
$$f(x;\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{x}{\sigma}-\mu)^2}{2}} = \frac{1}{\sigma} g\left(\frac{x}{\sigma}\right),$$
where $g$ is the density of $N(\mu, 1)$ or
$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2}}$$
and the desired conclusion follows by (2.2).

2. $X \in R(0, \theta)$. Then we take $x \in [0, 1]$ and get
$$P\left(\frac{X}{\theta} \leq x\right) = P(X \leq \theta x) = \frac{\theta x}{\theta} = x$$
as the distribution function of $R(0, \theta)$ is $F(z) = \frac{z}{\theta}$. The desired conclusion follows now from the first lemma. We have shown that $\frac{X}{\theta} \in R(0, 1)$, which is, however, immediate from the definitions.

3. Finally, we take the Weibull distribution, so that
$$F(x;\lambda) = \begin{cases} 1 - e^{-(x/\lambda)^k} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$
For $k = 1$ gives the exponential distribution, and $k = 2$ gives the Rayleigh distribution. It is obvious that $\lambda > 0$ is scale parameter.

Suppose that $X \in \mathrm{Exp}(\theta)$ is a lifetime measured in seconds. Then $\theta = E[X]$ is also measured in seconds. We might convert seconds to minutes. But the probability

$$P(X \leq x) = H\left(\frac{x}{\theta}\right),$$

is the same whether we measure in minutes or seconds, i.e., it is invariant with respect to scale or the units of measurement, as is quite reasonable.