The notation

\[ F(x; \theta) \]

denotes a distribution function that depends on a parameter \( \theta \). For example,

\[
F(x; \theta) = \begin{cases} 
1 - e^{-x/\theta} & \text{if } x \geq 0 \\
0 & \text{if } x < 0,
\end{cases}
\]
is the distribution function of the exponential distribution \( \text{Exp}(\theta) \).

\section{Definition of a Scale Parameter}

We define a scale parameter.

\textbf{Definition 1.1} Assume \( \theta > 0 \) in \( F(x; \theta) \). Then \( \theta \) is a scale parameter, if it holds for all \( x \) that

\[ F(x; \theta) = H\left(\frac{x}{\theta}\right), \tag{1.1} \]

where \( H(x) \) is a distribution function.

To take an example, \( \theta \) is scale parameter in the exponential distribution \( \text{Exp}(\theta) \), as we have

\[ F(x; \theta) = H\left(\frac{x}{\theta}\right), \]

where

\[
H(x) = \begin{cases} 
1 - e^{-x} & \text{if } x \geq 0 \\
0 & \text{if } x < 0.
\end{cases}
\]
2 Properties, Pivotal Variables

We observe two lemmas.

**Lemma 2.1** Assume $X \in F(x; \theta)$. $\theta$ is a scale parameter if and only if the distribution of

\[
\frac{X}{\theta}
\]

does not depend on $\theta$.

*Proof:* $\Rightarrow$: Assume $\theta$ is a scale parameter. Then the distribution of $\frac{X}{\theta}$ is given by

\[
P \left( \frac{X}{\theta} \leq z \right) = P \left( X \leq \theta z \right)
\]

since $\theta > 0$ by definition. Then, as we assume that $\theta$ is a scale parameter,

\[
P \left( X \leq \theta z \right) = F \left( \theta z; \theta \right) = H \left( \frac{\theta z}{\theta} \right) = H \left( z \right),
\]

or

\[
P \left( \frac{X}{\theta} \leq z \right) = H \left( z \right)
\]

which says that $\frac{X}{\theta}$ has distribution which does not depend on $\theta$.

$\Leftarrow$: We assume that $\frac{X}{\theta}$ has distribution, say $H$, which does not depend on $\theta > 0$. Then

\[
F(x; \theta) = P \left( X \leq x; \theta \right) = P \left( \frac{X}{\theta} \leq \frac{x}{\theta}; \theta \right)
\]

since $\theta > 0$. But by assumption

\[
P \left( \frac{X}{\theta} \leq \frac{x}{\theta}; \theta \right) = H \left( \frac{x}{\theta} \right)
\]

where the distribution function $H \left( z \right)$ does not depend on $\theta$. But then we have for any $x$ shown that

\[
F(x; \theta) = H \left( \frac{x}{\theta} \right)
\]

which in view of (1.1) gives the claim as asserted. □

This lemma says that $\frac{X}{\theta}$ is a *pivotal variable*. 

2
Remark 2.1 Note that a pivotal variable need not be a statistic — the variable can depend on parameters of the model, but its distribution must not. If it is a statistic, then it is known as an ancillary statistic. Pivotal quantities are used to the construction of test statistics, e.g., Student's t-statistic is pivotal for a normal distribution with unknown variance (and mean). Pivotal variables provide in addition a method of constructing confidence intervals, and the use of pivotal quantities improves performance of the bootstrap, as defined later.

Lemma 2.2 Assume $F(x; \theta)$ has a density $\frac{d}{dx} F(x; \theta) = f(x; \theta)$ for all $x$. Then $\theta$ is a scale parameter if and only if
\begin{equation}
    f(x; \theta) = \frac{1}{\theta} g \left( \frac{x}{\theta} \right),
\end{equation}
where $g$ is a probability density.

Proof: $\Rightarrow$: If $F(x; \theta)$ has an integrable derivative and $\theta$ is a scale parameter, then it holds from (1.1) that
\[
    f(x; \theta) = \frac{d}{dx} F(x; \theta) = \frac{d}{dx} H \left( \frac{x}{\theta} \right) = \frac{1}{\theta} g \left( \frac{x}{\theta} \right)
\]
where we have set $g(x) = \frac{d}{dx} H(x)$, which is a density function, as $H$ is a distribution function.

$\Leftarrow$: We assume that
\[
    f(x; \theta) = \frac{1}{\theta} g \left( \frac{x}{\theta} \right).
\]
Then
\[
    F(x; \theta) = \int_{-\infty}^{x} f(u; \theta) du = \frac{1}{\theta} \int_{-\infty}^{x} g \left( \frac{u}{\theta} \right) du.
\]
Here we make a change of variable $t = \frac{u}{\theta}$, $du = \theta dt$ and thus
\[
    \frac{1}{\theta} \int_{-\infty}^{x} g \left( \frac{u}{\theta} \right) du = \int_{-\infty}^{\frac{x}{\theta}} g \left( t \right) dt = G \left( \frac{x}{\theta} \right),
\]
where $\frac{d}{dx} G(x) = g(x)$. In other words we have obtained for every $x$ that
\[
    F(x; \theta) = G \left( \frac{x}{\theta} \right),
\]
and since $G(x)$ is a distribution function, the desired assertion follows by (1.1).

3 Examples and the Scale-Free property

We give first some further examples.

1. $X \in N(\mu \sigma, \sigma^2)$. We guess that $\sigma$ is a scale parameter. The pertinent density is

$$f(x; \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and some elementary algebra gives

$$f(x; \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma} g\left(\frac{x}{\sigma}\right),$$

where $g$ is the density of $N(\mu, 1)$ or

$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2}}$$

and the desired conclusion follows by (2.2).

2. $X \in R(0, \theta)$. Then we take $x \in [0, 1]$ and get

$$P\left(\frac{X}{\theta} \leq x\right) = P(X \leq \theta x) = \frac{\theta x}{\theta} = x$$

as the distribution function of $R(0, \theta)$ is $F(z) = \frac{z}{\theta}$. The desired conclusion follows now from the first lemma. We have shown that $\frac{X}{\theta} \in R(0, 1)$, which is, however, immediate from the definitions.

3. Finally, we take the Weibull distribution, so that

$$F(x; \lambda) = \begin{cases} 1 - e^{-x/\lambda^k} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

For $k = 1$ gives the exponential distribution, and $k = 2$ gives the Rayleigh distribution. It is obvious that $\lambda > 0$ is scale parameter.
Suppose that $X \in \text{Exp}(\theta)$ is a lifetime measured in seconds. Then $\theta = E[X]$ is also measured in seconds. We might convert seconds to minutes. But the probability

$$P(X \leq x) = H\left(\frac{x}{\theta}\right),$$

is the same whether we measure in minutes or seconds, i.e., it is invariant with respect to scale or the units of measurement, as is quite reasonable.