Convergence theorems

Thm 1 *(Fatou's Lemma)*

Let $\{f_n\}_{n=1}^{\infty}$ *of measurable functions such that*

$$f_n \geq 0, \qquad n = 1, 2, \ldots$$

and

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \text{ in } X.$$

Then

$$\int_X f(x) d\mu(x) \leq \liminf_n \int_X f_n(x) d\mu(x).$$

Thm 2 *(Monotone convergence)*

Let $\{f_n\}_{n=1}^{\infty}$ *of measurable functions such that*

$$f_n \geq 0, \qquad n = 1, 2, \dots$$

and

$$f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots \qquad \text{for all } x.$$

Define f *by*

$$f(x) = \lim_n f_n(x).$$

Then

$$\int_X f(x) d\mu(x) = \lim_n \int_X f_n(x) d\mu(x).$$

Thm 3 *(Dominated convergence)*

Let $\{f_n\}_{n=1}^{\infty}$ *of measurable functions such that*

$$f_n(x) \to f(x)$$

and suppose that there exists an integrable function g *such that*

$$|f_n(x)| \leq g(x) \qquad \textit{for all } n \textit{ and all } x.$$

Then

$$\int_X f(x) d\mu(x) = \lim_n \int_X f_n(x) d\mu(x).$$

Ideally you want the Lesbesgue measure m on $\mathbb{R}$ to have the following properties:

1. $m(A)$ should be defined for every set $A \in 2^{\mathbb{R}}$

2. For an interval I, $m(I) =$ length of I.

3. If $A_1, A_2, \ldots$ are disjoint sets for which m is defined, then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n).$$

4. m is translation invariant; that is, if A is a set for which m is defined and if $A + y = \{x + y | x \in A\}$, then

$$m(A + y) = m(A).$$

Cannot construct a measure having all four of these properties.

It is not known whether there is a set function satisfying the first three.

Therefore we must weaken at least one of the properties, and usually one chooses to keep the last three, and weaken the first so that m is defined for as many sets as possible (ending up with the Lesbesgue measure on the Borel algebra).

It is also possible to replace property 3 of countable additivity by the weaker property of finite additivity or by the property of countable subadditivity.