Conditional expectation

Def: We have

$$E[X|\mathcal{F}] = Z$$

if Z is a random variable such that

1. Z is $\mathcal{F}$-measurable

2. it holds that

$$\int_F Z dP = \int_F X dP$$

   for all $F \in \mathcal{F}$.

Properties of the conditional expectation:

1. $X \geq 0 \Longrightarrow E[X|\mathcal{F}] \geq 0 \quad P - a.s.$

2. $E[\alpha X + \beta Y|\mathcal{F}] = \alpha E[X|\mathcal{F}] + \beta E[Y|\mathcal{F}]$

3. If $\mathcal{G} \subseteq \mathcal{F}$ then

$$E[E[X|\mathcal{F}]|\mathcal{G}] = E[X|\mathcal{G}]$$

and especially

$$E[E[X|\mathcal{G}]] = E[X].$$

Also

$$E[E[X|\mathcal{G}]|\mathcal{F}] = E[X|\mathcal{G}]$$

4. If X is $\mathcal{F}$-measurable then

$$E[X|\mathcal{F}] = X \quad P - a.s.$$

and

$$E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$$

5. If X is independent of $\mathcal{F}$ then

$$E[X|\mathcal{F}] = E[X]$$

6. If f is convex then

$$f(E[X|\mathcal{F}]) \leq E[f(X)|\mathcal{F}] \quad P - a.s.$$

Relation between $\int_\Omega X(\omega)dP(\omega)$ and $\int_{\mathbb{R}} x f_X(x)dx$?

Recall that $X : \Omega \longrightarrow \mathbb{R}$

The **distribution measure** μ_X is defined by

$$\mu_X(B) = P(\{\omega \in \Omega | X(\omega) \in B\}) \quad B \in \mathcal{B}$$

The (cumulative) **distribution function** F_X is defined by

$$F_X(x) = P(\{\omega \in \Omega | X(\omega) \leq x\})$$

By definition

$$E[g(X)] = \int_{\Omega} g(X(\omega))dP(\omega)$$

but it can be shown that

$$E[g(X)] = \int_{\mathbb{R}} g(x)d\mu_X(x) = \int_{\mathbb{R}} g(x) \underbrace{dF_X(x)}_{f_X(x)dx}$$