Cauchy sequences and complete spaces

Recall that

$$\lim_{n \to \infty} f_n = f$$

means that for every $\varepsilon > 0$ there exists an N such that

$$\|f_n - f\| < \varepsilon \quad \text{for all } n \geq N.$$

Problem: To check for convergence you need to know what the limit is!

The remedy for complete spaces is

Def 1 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *is a* **Cauchy sequence** *if for every* $\varepsilon > 0$ *there exists an* N *such that*

$$\|f_n - f_m\| < \varepsilon \quad \text{for all } m, n \geq N.$$

Def 2 *A normed linear space is called* **complete** *if every Cauchy sequence in the space converges, i.e. for each Cauchy sequence* $\{f_n\}_{n=1}^{\infty}$ *in the space there is an element* f *in the space such that*

$$f_n \to f.$$

Advantage: Now you can check for convergence without knowing the limit!

Examples of complete spaces:

- $\mathbb{R}$ with $\|f\| = |f|$.

- $L^1(\Omega, \mathcal{F}, P)$ with

$$\|X\| = \int_\Omega |X| dP.$$

- $L^2(\Omega, \mathcal{F}, P)$ with

$$\|X\| = \sqrt{\int_\Omega |X|^2 dP}.$$