SF2972 GAME THEORY
Lecture 8

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Today we will consider *finite* normal-form games $G = \langle N, S, u \rangle$, that is, games with finitely many players and with finitely many strategies for each player:

1. $N = \{1, ...n\}$ is the finite set of players

2. $S = \times_{i \in N} S_i$ is the finite set of pure strategy profiles

3. $u : S \to \mathbb{R}^n$ is the combined payoff function

   $u_i(s) \in \mathbb{R}$ being the payoff/utility to player i when strategy profile $s = (s_1, ..., s_n)$ is played

1 Mixed strategies

1.1 Geometry

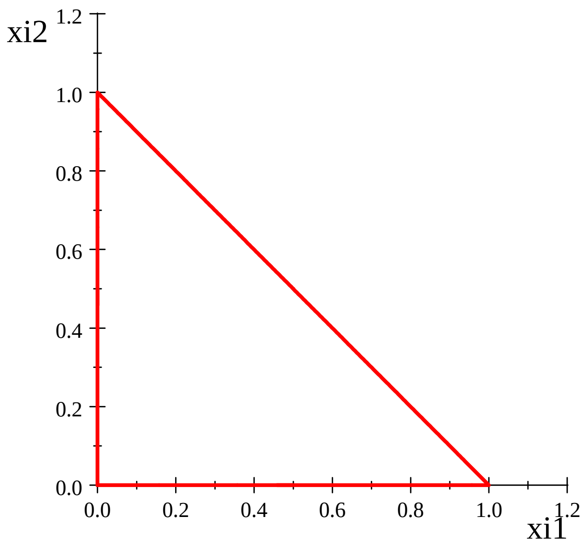
- Let $S_i = \{1, ..., m_i\}$ be i's pure strategies

- The player's *mixed-strategy simplex:*

$$X_i = \Delta_i = \Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{h=1}^{m_i} x_{ih} = 1\}$$

- The *vertices* of Δ_i are the *unit vectors,* $e_i^1, ..., e_i^{m_i} \in \mathbb{R}_+^{m_i}$

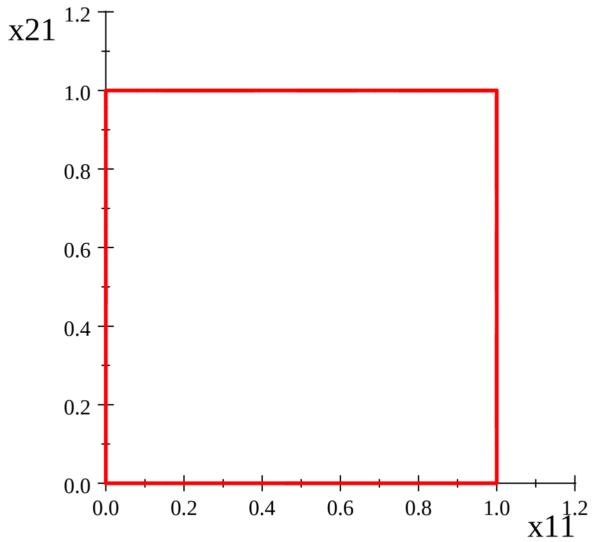
- Example: $|S_i| = 3$



- The *mixed-strategy polyhedron:*

$$X = \Box = \Box(S_i) = \times_{i \in N} \Delta(S_i)$$

- Example: $n = |S_1| = |S_2| = 2$



1.2 Mixed-strategy payoff functions

The payoff to a player, when mixed strategies are used ,is defined as the (mathematical) expectation of the player's payoff:

Definition 1.1 *The payoff function for each player* $i \in N$, $\tilde{u}_i : \square(S) \to \mathbb{R}$, *is defined by*

$$\tilde{u}_i(x) = \sum_{s \in S} \left(\prod_{i=1}^{n} x_{i,s_i} \right) u_i(s)$$

- Note that this is a polynomial function that is linear in each player's randomization. In particular, it is linear in the player's own mixed strategy

$$\tilde{u}_i(x'_i, x_{-i}) = \sum_{h \in S_i} \tilde{u}_i(e_i^h, x_{-i}) \cdot x'_{ih} \qquad \forall x'_i \in \Delta(S_i)$$

1.3 Interpretations

[Osborne and Rubinstein 3.2]

1. Intentional randomization (the rationalistic interpretation)

2. Population frequencies (the mass-action interpretation)

3. Mixed strategies as (others') beliefs, not (your) actions

2 Best replies and dominance relations

Definition 2.1 *The* mixed-strategy extension *of a finite game* $G = \langle N, S, u \rangle$ *is the game* $\tilde{G} = \langle N, \square(S), \tilde{u} \rangle$.

2.1 Best replies

- The i:th player's pure-strategy best-reply correspondence on the polyhedron of mixed-strategy profiles, $\beta_i : \square(S) \rightrightarrows S_i$, is defined by

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(e_i^h, x_{-i}) \geq \tilde{u}_i(e_i^k, x_{-i}) \; \forall k \in S_i\}$$

- Mixed strategies cannot give higher payoffs than pure (why?):

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(e_i^h, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i}) \; \forall x_i' \in \mathbf{\Delta}_i\}$$

Definition 2.2 *The i:th player's mixed-strategy best-reply correspondence $\tilde{\beta}_i$: $\square(S) \rightrightarrows \boldsymbol{\Delta}_i$ is defined by*

$$\tilde{\beta}_i(x) = \{x_i^* \in \boldsymbol{\Delta}_i : \tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i}) \ \forall x_i' \in \boldsymbol{\Delta}_i\}$$

- Note that

$$\tilde{\beta}_i(x) = \{x_i^* \in \boldsymbol{\Delta}_i : \textit{supp}(x_i^*) \subset \beta_i(x)\}$$

- $\tilde{\beta}_i(x)$ is a *face* (or subsimplex) of the simplex $\boldsymbol{\Delta}_i$

- The *combined* **mixed** *BR correspondence* $\tilde{\beta} : \square(S) \rightrightarrows \square(S)$ is defined by

$$\tilde{\beta}(x) := \times_{i\in N}\tilde{\beta}_i(x)$$

Definition 2.3 *A mixed-strategy profile x is a* **Nash equilibrium** *of $\tilde{G} = \langle N, \square(S), \tilde{u}\rangle$ if $x \in \tilde{\beta}(x)$*

2.2 Dominance relations

Definition 2.4 $x_i^* \in \Delta_i$ **strictly dominates** $x_i' \in \Delta_i$ *if*

$$\tilde{u}_i(x_i^*, x_{-i}) > \tilde{u}_i(x_i', x_{-i}) \ \forall x \in \Box$$

Definition 2.5 $x_i^* \in \Delta_i$ **weakly dominates** $x_i' \in \Delta_i$ *if*

$$\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i}) \ \forall x \in \Box \text{ with } > \text{ for some } x \in \Box$$

Definition 2.6 *A strategy that is not weakly dominated is* **undominated.**

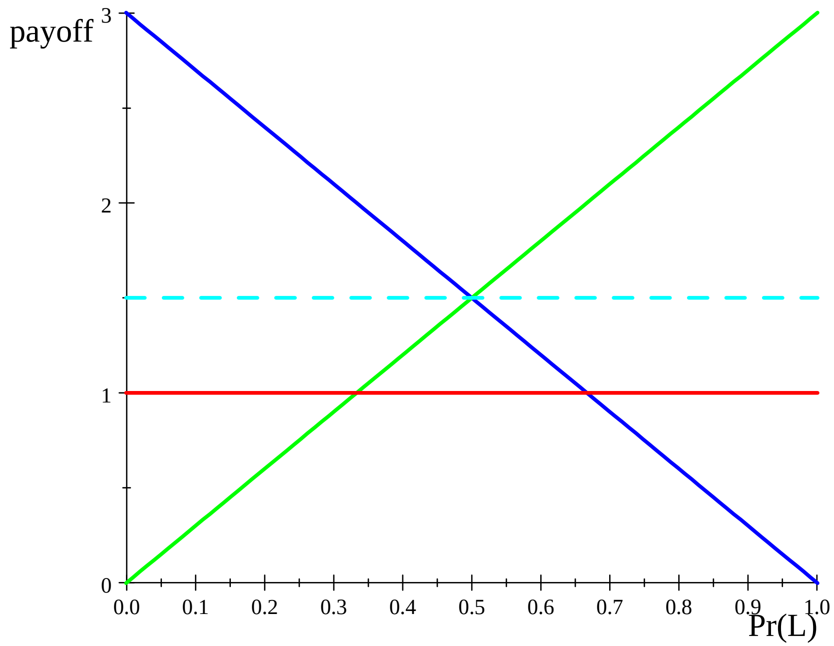
1. For a player to use a strictly dominated strategy is irrational: is not optimal under any belief

2. To use a weakly dominated strategy is like not taking an insurance that is available for free, an insurance against all eventualities associates with all *other* players' actions. In simultaneous-move games, it seems unwise not to take such an insurance.

3. A strategy can be strictly dominated without being (weakly or strictly) dominated by any *pure* strategy

Example 2.1 *Consider player 1 with payoff matrix*

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}$$



- In arbitrary finite games: *iterated elimination* of strictly dominated pure strategies

  (a) Halts after a finite number of rounds

  (b) End-result independent of order of elimination

  (c) Nash equilibria never eliminated

Example 2.2 *A two-player game with payoff bi-matrix*

$$(A, B) = \begin{bmatrix} 3,3 & 0,0 & 6,1 \\ 0,0 & 0,0 & 5,2 \\ 1,6 & 2,5 & 4,4 \end{bmatrix}$$

2.3 Dominance vs. best replies

- Pure best replies are clearly not strictly dominated

Q1: If a pure strategy is *not* strictly dominated, is it then a best reply to *some* (mixed-)strategy profile?

Proposition 2.1 (Pearce, 1984) *Suppose* $n = 2$. *Then*

(a) $h \in \beta_i(x)$ *for some* $x \in \Box \quad \Leftrightarrow \quad h \in S_i$ *not strictly dominated*

(b) $h \in \beta_i(x)$ *for some* $x \in int(\Box) \quad \Leftrightarrow \quad h \in S_i$ *undominated*

- What about games with more than two players?

3 Rationalizability

[Osborne and Rubinstein 4.1-4.2]

- Consider a finite game in normal form, $G = \langle N, S, u \rangle$ and assume

  A1 (*Rationality)*: Each player i forms a probabilistic belief $\mu_j^i \in \Delta\left(S_j\right)$ about every other player j's pure strategy, a belief that does not contradict any information or knowledge that player i has, and player i chooses a (pure or mixed) strategy that maximize his or her expected payoff, assuming statistical independence between other player's strategy choices [Osborne and Rubinstein 5.1.2]

  A2 (*Common knowledge*): The game G and the players' rationality (A1) is *common knowledge* among the players [Osborne and Rubinstein 5.2]

- In Lecture 1, we observed that [A1 $\wedge$ A2] $\not\Rightarrow$ NE

Q1: What does A1 and A2 then imply (if anything)?

A1: Rationalizability!

Q2: What is, then, "rationalizability"?

A2: The definition is recursive and a bit involved. We make it in steps.

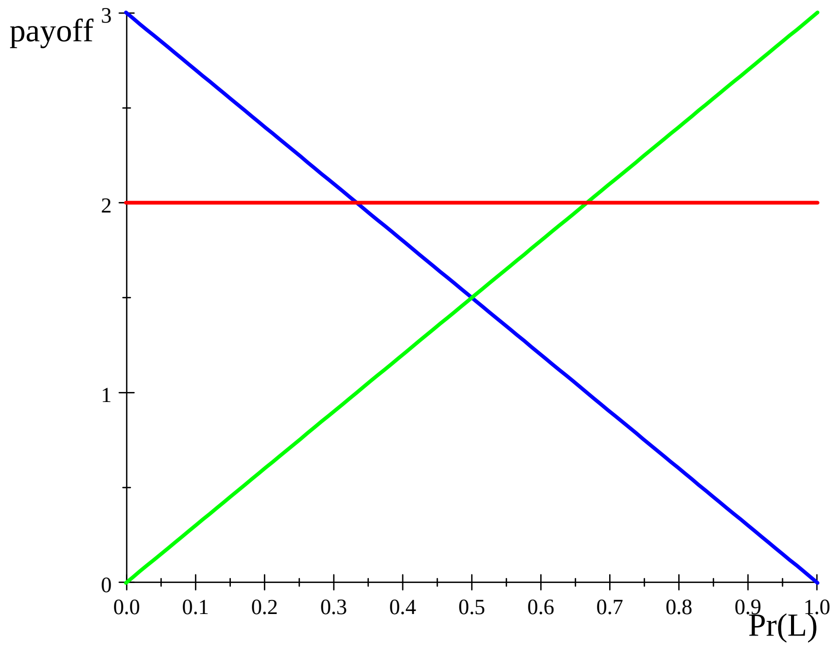
- For any $X = \times_{j=1}^{n} X_j$, where each $X_j \subset \Delta\left(S_j\right)$, write

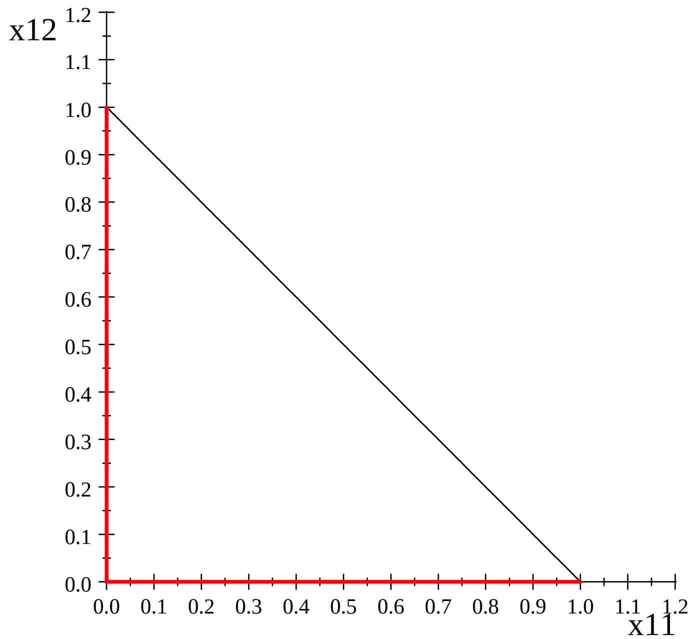
$$\tilde{\beta}_i(X) = \left\{x_i^* \in \Delta(S_i) : x_i^* \in \tilde{\beta}_i(x) \text{ for some } x \in X\right\}$$

- Note that the set $\tilde{\beta}_i(X)$ is not necessarily convex even if X is convex

Example 3.1 *Consider player 1 with payoff matrix*

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$$




x12
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2
x11

- Let $C^0 = \square(S)$ and define the sequence $\left\langle C^t \right\rangle_{t\in\mathbb{N}}$ recursively by

$$\begin{cases} C_i^{t+1} = conv\left[\tilde{\beta}_i\left(C^t\right)\right] \;\; \forall i \in N \\ C^{t+1} = \times_{i\in N} C_i^{t+1} \end{cases}$$

- Here "conv" means "convex hull of"

Definition 3.1 *The convex hull of a set $X \subset \mathbb{R}^n$ is the intersection of all convex sets that contain X.*

Definition 3.2 (Pearce, 1984) *A strategy* $x_i \in \Delta(S_i)$ *is* **rationalizable for player** i *if* $x_i \in C_i^{\infty}$*, where*

$$C_i^{\infty} = \cap_{t \in \mathbb{N}} C_i^t.$$

Proposition 3.1 $C_i^{\infty} = \Delta(Z_i)$ *for a non-empty subset* $Z_i \subset S_i$

Proof: For any given player $i \in N$:

1. $\forall t$: C_i^t is a subsimplex of $\Delta(S_i)$
2. $\forall t$: $C_i^{t+1} \subseteq C_i^t$
3. The collection of subsimplices of $\Delta(S_i)$ is finite

Definition 3.3 *A pure strategy* $h \in S_i$ *is* **rationalizable** *if* $h \in Z_i$.

- Reconsider earlier examples

- Note that (the support of) any NE is rationalizable

3.1 Rationalizability vs. iterated strict dominance

Discussion in class

- First consider two-player games
- Then consider games with more players
- Osborne's and Rubinstein's definition

4 Evolutionary stability

A population scenario

1. A large population of individuals who are recurrently and randomly matched in pairs to play a symmetric and finite two-player game

2. Initially, all individuals always use the same pure or mixed strategy, x

3. Suddenly, a small population share switch to strategy y

4. If those who play x on average do better than those who play y, then x is *stable against* y

5. x is *evolutionarily stable* if it is stable against *all* $y \neq x$

- The domain of the analysis now restricted to symmetric and finite two-player games

Definition 4.1 *A finite two-player game G is* **symmetric** *if $S_1 = S_2$ and $u_2(h,k) = u_1(k,h)$ for all pure strategies h and k.*

- Payoff bimatrix (A, B) such that $B = A^T$

- Write S for $S_1 = S_2$ and Δ for $\Delta(S)$, the mixed-strategy simplex:

$$\Delta = \{x \in \mathbb{R}_+^m : \sum_{i \in S} x_i = 1\}$$

- Write the payoff to any strategy $x \in \Delta$, when used against any strategy $y \in \Delta$ as

$$u(x, y) = x \cdot Ay$$

Example 4.1 (Prisoners' dilemma) *symmetric*

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

Example 4.2 (Matching-pennies) *asymmetric*

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

Example 4.3 (Coordination) *doubly symmetric*

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

Definition 4.2 $x \in \Delta$ *is an* **evolutionarily stable strategy (ESS)** *if for every strategy* $y \neq x$ $\exists$ $\bar{\varepsilon} \in (0,1)$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon})$*:*

$$u\left[x, \varepsilon y + (1-\varepsilon)x\right] > u\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- Let $\Delta^{ESS} \subset \Delta$ denote the (sometimes empty) set of ESSs

Proposition 4.1 $x \in \Delta^{ESS}$ *if and only if*

$$u(x,x) \geq u(y,x) \qquad \forall y \in \Delta$$

and

$$u(y,x) = u(x,x) \Rightarrow u(y,y) < u(x,y)$$

- Hence: $x \in \Delta^{ESS} \quad \Rightarrow \quad (x,x)$ Nash equilibrium

Example 4.4 (PD)

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

$$\Delta^{ESS} = \{D\}$$

Example 4.5 (CO)

$$\begin{array}{ccc} & A & B \\ A & 2,2 & 0,0 \\ B & 0,0 & 1,1 \end{array}$$

$$\Delta^{ESS} = \{A, B\}$$

Example 4.6 (Hawk-Dove)

$$\begin{array}{ccc} & H & D \\ H & -1, -1 & 4, 0 \\ D & 0, 4 & 2, 2 \end{array}$$

$$\Delta^{ESS} = ?$$

THE END