SF2972 GAME THEORY
Problem Set 1
Due January 27, at the lecture

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1. Find all Nash equilibria, in pure and mixed strategies, in each of the three games in Lecture 1, Section 3.4.

   Solution: *CO-game:* (A, A), (B, B) *and* $((1/3, 2/3), (1/3, 2/3))$. *MP-game:* $((1/2, 1/2), (1/2, 1/2))$. *Third game:* (M, C).

2. Find all Nash equilibria, in pure and mixed strategies, in the partnership game in Lecture 1, Section 6.3.

   Solution: (W, S), (S, W) *and* $((1/2, 1/2), (1/2, 1/2))$.

3. Write up the normal form of Games 1-4 in Lecture 2 (Section 1). For each of these normal-form games, find all Nash equilibria, in pure and mixed strategies.

   Solution:

Game 1:

	aa	ab	ba	bb
A	$3,1$	$3,1$	$0,0$	$0,0$
B	$0,0$	$1,3$	$0,0$	$1,3$

Strategy ba *is strictly dominated for player 2, hence not used in NE. Pure NE:* (A, aa), (A, ab) *and* (B, bb). *Mixed NE:* (A, x_2) *where* x_2 *is any randomization between* aa *and* ab. (B, x_2) *where* x_2 *is any randomizataion between* ab *and* bb *that attaches probability* $\leq 1/3$ *to* ab.

Game 2:

	a	b
A	$3,1$	$0,0$
B	$0,0$	$1,3$

NE: (A, a), (B, b) *and* $((3/4, 1/4), (1/4, 3/4))$.

Game 3:

	aa	ab	ba	bb
A	$3,0$	$3,0$	$0,1$	$0,1$
B	$0,3$	$1,0$	$0,3$	$1,0$

NE: (A, ba), (B, ba) *and* (x_1, ba) *for all mixed strategies* x_1.

Game 4:

	C	F
A	$1,3$	$1,3$
E	$2,2$	$0,0$

NE: (A, F), (E, C) *and* (A, x_2) *for all* x_2 *that assigns probability* $\geq 1/2$ *to* F.

4. Consider the following 2×2 normal-form game G, for arbitrary $a, b > 0$:

$$\begin{array}{ccc} & H & T \\ H & a, 0 & 0, b \\ T & 0, b & 1, 0 \end{array}$$

(a) Find all pure and mixed Nash equilibria in G.

Solution *(to both a and b): Let player 1 (2) play H with probability* x (y). *For player 1:* $H \succcurlyeq T$ *iff* $y \geq 1/(1+a)$. *For player 2:* $H \succcurlyeq T$ *iff* $x \geq 1/2$. *Draw a diagram of the unit square and identify the NE with the intersections of the two best-reply graphs.*

(b) For each pure or mixed Nash equilibrium in G, and each player, find the player's set of pure and mixed best replies to the equilibrium in question.

5. Reconsider the Cournot oligopoly game in Lecture 2, Section 6.1. Let there be n firms in the market, where n is an arbitrary positive integer (including the case $n = 1$). For each firm, let its strategy set be $[0, 100]$, let $Q = q_1, ..., q_n$ and let the payoff function of each firm be its profit, defined as

$$\pi_i(q_1, ..., q_n) = \begin{cases} (100 - Q) \cdot q_i & \text{if } Q \leq 100 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i = 1, 2, ...n$$

(a) Which of the hypotheses in Theorem 7.1 in Lecture 2 are met, and which are not?

Solution: *All hypotheses are met.*

(b) Find a Nash equilibrium (in pure strategies), for each $n \in \mathbb{N}$. Is it unique?

Solution: *If all* $q_i > 0$ *and* $Q < 100$, *then a necessary and sufficient F.O.C. for NE is* $100 - Q - q_i = 0 \quad \forall i$. *Hence,* $q_1 = q_2 = ... = q_n = q^*$, *where* $q^* = 100/(n+1)$. *This is the only NE when* $n = 1$. *However, for* $n > 1$, *there are other NE, namely when* $\sum_{j \neq i} q_j \geq 100$ *for all* i.

(c) Compute aggregate supply, in (your) Nash equilibrium, $Q^{NE}(n)$, as a function of n. Explain its dependence on n, the number of firms in the market, and explain also how each firm's output depends on n.

Solution: *In the unique symmetric NE:* $Q^{NE}(n) = 100n/(n+1) \rightarrow 100$ *as* $n \rightarrow +\infty$.

6. Two individuals, 1 and 2, contribute to a public good (say, a clean shared kitchen) by making individual efforts $x \in [0, 1]$ and $y \in [0, 1]$. The resulting level of the public good is $x + y$. Individual utilities are given by

$$u_1(x, y) = (x + y)\, e^{-x} \quad \textit{and} \quad u_2(x, y) = (x + y)\, e^{-y}$$

Each individual strives to maximize his or her expected utility.

(a) Game A: Suppose both effort levels are chosen simultaneously. Which of the hypotheses in Theorem 7.1 in Lecture 2 are met, and which are not? Does this game have a pure-strategy Nash equilibrium? Find all its pure-strategy Nash equilibria.

Solution: *All hypotheses, expect compactness, are met. Necessary and sufficient F.O.C for NE with* $x, y > 0$ is $x + y = 1$. *Hence, all such strategy pairs are NE. Also* $(x, y) = (0, 1)$ *and* $= (1, 0)$ *are NE.*

(b) Game B: Suppose individual 1 first chooses her effort level, and that this is observed by individual 2, who then chooses his effort level. Solve it by backward induction.

Solution: *For any* $x \in [0, 1]$*, player 2 will choose* $y = 1 - x$*. Hence, player 1 will choose* x *so as to maximize* $(x + 1 - x) e^{-x} = e^{-x}$*. Thus,* $x = 0$ *and* $y = 0$ *is the unique backwards-inductive solution (subgame perfect equilibrium outcome).*

(c) Write up the normal form of Game B. Prove that your backward-inductive solution constitutes a Nash equilibrium. Show that this game has infinitely many pure-strategy Nash equilibria. In particular, show that there exist a Nash equilibrium in which individual 2 makes effort $x_2 = 1/2$.

Solution: *The strategy set of player 1 is the unit interval,* $X = [0, 1]$*, and that of player 2 is the set* F *of functions from the unit interval to itself. Let 2's strategy be the step function* $f_a(x) = 1 - x$ *if* $x \geq a$ *and* $f(x) = 0$ *if* $x < a$*. The strategy pair* (a, f_a) *is a NE for all* $a \in [0, 1]$.