SF2972 GAME THEORY
Solutions to Problem Set 3

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1. Consider the price competition between two firms with identical products. Each firm i has a constant unit cost c_i of production, where $0 \leq c_i \leq 20$, and demand is given by $D(p) = \max\{0, 100 - p\}$. Each firm chooses its price $p_i \in [0, 100]$ so as to maximize its profit

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c_i) D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c_i) D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \qquad (\text{for } j \neq i)$$

(a) Suppose first that one of the firms would be a monopolist (the other firm being absent). What would be the (optimal) price for firm i in such a monopoly situation?

Solution: $p^m \in \arg\max_{0 \leq p \leq 100} (p - c_i)(100 - p)$. *F.O.C:* $100 - p - (p - c_i) = 0$. *Hence:* $p^m = 50 + c_i/2$.

(b) Suppose that both firms are in the market and $c_1 = c_2 = c$. Find the unique (pure-strategy) Nash equilibrium price and compare with the monopoly price at that unit cost.

Solution: *Unique NE* $p_1 = p_2 = c \leq 20$. *This is lower than* $p^m = 50 + c/2$.

(c) Is the Nash equilibrium price in (b) undominated?

Solution: *No.*

(d) Does a (pure-strategy) Nash equilibrium exist if $c_1 \neq c_2$?

Solution: *No. [Assume that* $0 \leq c_1 < c_2 \leq 20$*. Then* $p_2 \geq c_2$ *in NE, and in fact* $p_2 = c_2$*. But then no best reply* p_1 *exists.]*

(e) Do (b)-(d) in the case of a smallest monetary unit: $c_i, p_i \in V = \{1, 2, ..., 99, 100\}$ for $i = 1, 2$.

Solution: *(b) Two NE,* $(p_1, p_2) = (c, c)$ *and* $(p_1, p_2) = (c + 1, c + 1)$*. (c) The first NE is weakly dominated, not the second. (d)* $(p_1, p_2) = (c_2 - 1, c_2)$.

2. Consider two firms that sell the same product, but at locations x and y in some set $X \subset [0, 1]$. Both firms have zero production costs. Consumers are uniformly distributed on $L = [0, 1]$, and each consumer buys 1 unit from the nearest seller (they split the market evenly if $x = y$). Normalize the total consumer population to 1. Each firm strives to maximize its profit (its market share times its price).

(a) Suppose first that both firms sell at the same fixed price $p > 0$. Write this up as a normal-form game and find its unique Nash equilibrium, (x^*, y^*), when $X = L$.

Solution: $S_1 = S_2 = [0,1]$, $\pi_1(x,y) = (x+y)/2$ *if* $x < y$, $\pi_1(x,y) = 1/2$ *if* $x = y$ *etc. and* $x^* = y^* = 1/2$.

(b) Do (a) when $X \subset [0,1]$ is an arbitrary set such that $1/2 \in X$.

Solution: $S_1 = S_2 = X$, *payoffs and NE as in (a).*

(c) Suppose now that each firm chooses its price after they both have chosen locations. More exactly: In stage 1 both firms simultaneously choose locations, $x \in X$ and $y \in X$. In stage 2, both firms observe each others' locations and simultaneously choose their prices, $p_1 \geq 0$ and $p_2 \geq 0$. A consumer located at any point $z \in L = [0,1]$ buys from firm 1 if

$$p_1 + \tau \cdot |x - z| < p_2 + \tau \cdot |y - z|$$

and from firm 2 if the inequality is reversed, where $\tau > 0$ is a (transportation cost) parameter. In case of equality, the consumer buys with equal probability from any one of the two firms. Each firm strives to maximize its (expected) profit.

(c1) What is the strategy set of each firm in this two-stage game? Define the firms' payoff functions [their profits as functions of their strategies].

Solution: $S_1 = S_2 = X \times F$, *where* F *is the set of functions from* X^2 *to* R_+. *If* $s_1 = (x,f)$ *and* $s_2 = (y,g)$, *then* $p_1 = f(x,y)$ *and* $p_2 = g(x,y)$. *Let*

$$z^0 = \frac{x+y}{2} + \frac{p_2 - p_1}{2\tau}$$

If $x < y$ *and* $|p_1 - p_2| < \tau \cdot (x - y)$: $\pi_1(s) = p_1 \cdot z^0$ *and* $\pi_2(s) = p_2 \cdot \left(1 - z^0\right)$. *If* $x < y$ *and* $p_1 - p_2 < \tau \cdot |x - y|$: $\pi_1(s) = p_1$ *etc.*

(c2) Let $X = \{1/4, 1/2, 3/4\}$. For what parameter values $\tau > 0$, if any, is it a Nash equilibrium outcome that they locate at $x = 1/4$ and $y = 3/4$ and choose the same price? [Hint: the firms may have to make (non-credible) price-threats against each others' alternative locations.]

Solution: *Let* $s_1^* = (1/4, f)$ *and* $s_2^* = (3/4, f)$ *where* $f(1/4, 3/4) = p > 0$ *and* $f(x,y) = 0$ *otherwise. [Each firm threatens to price at zero if the other firm chooses another location.] Then* $\pi_1(s^*) = \pi_2(s^*) = p/2$. *This is a NE if* $p = \tau$, *since then (i) no price-undercutting, at the given locations, is profitable, (ii) a move to the location* $1/2$, *and choosing any price* p' *there, results in profits* $(3/8 - p'/(2\tau))\,p' < \tau/2$ *(the other firm will respond by pricing at zero), and (iii) a move to the other firm's location results in zero profits since the other firm then will price at zero.*

(c3) Let $X = \{1/4, 1/2, 3/4\}$. For what parameter values $\tau > 0$, if any, is it a subgame-perfect equilibrium outcome that they locate at $x = 1/4$ and $y = 3/4$ and set the same price? What is the range of subgame perfect equilibrium prices at those locations? [Hint: solve by backward induction, by first considering price competition at all possible location pairs.]

Solution: *In comparison with (c2), we now impose the additional requirement that the two firms' prices constitute a NE for all possible locations* *of the firms. Now let* $s_1^* = (1/4, f)$ *and* $s_2^* = (3/4, f)$ *for some price function* f*. Such a strategy profile* s^* *is a SPE if* $(p,p) = (f(x,y), f(x,y))$ *is a NE* $\forall x, y \in X$*. This requires* $f(x,x) = 0$ $\forall x \in X$ *(zero price when the firms are at the same location). From (c2), we necessarily have* $f(1/4, 3/4), f(3/4, 1/4) \leq \tau/2$*. It remains to identify the NE (identical) constraints on* $f(1/4, 1/2)$ *and* $f(1/4, 1/2)$*, that is, the NE prices when one of the firms locate in the middle.*

3. Consider two firms competing in a homogeneous product market. Firm 1 has unit production cost $c = 10$. Firm 2 has either unit production cost c_L or c_H where $0 \leq c_L < c_H \leq 20$. The probabilities are $\Pr[c_L] = \lambda$ and $\Pr[c_H] = 1 - \lambda$, where $\lambda \in (0,1)$. Both firms know the cost of Firm 1, and Firm 2 knows its own cost, but Firm 1 is not informed of 2's cost. Both firms simultaneously select output levels, q_1 and q_2, in $[0, 100]$. The market clears at the price $p = \max\{0, 100 - q_1 - q_2\}$. Formalize this as a three-player game (between Firm 1, Firm 2L and Firm 2H) and solve for Nash equilibrium.

   Solution: *Let* $q = (q_1, q_{2L}, q_{2H})$*. The payoff functions for the three players:*

$$\begin{cases} \pi_1(q) = \lambda \cdot [\max\{0, 100 - q_1 - q_{2L}\} - c] \cdot q_1 \\ \quad + (1-\lambda) \cdot [\max\{0, 100 - q_1 - q_{2H}\} - c] \cdot q_1 \\ \pi_{2L}(q) = [\max\{0, 100 - q_1 - q_{2L}\} - c_L] \cdot q_{2L} \\ \pi_{2H}(q) = [\max\{0, 100 - q_1 - q_{2H}\} - c_H] \cdot q_{2H} \end{cases}$$

   Necessary F.O.C.s for an NE with $q > 0$ *and* $q_1 + q_{2L}, q_1 - q_{2H} < 100$:

$$\begin{cases} 100 - 2q_1 - \lambda q_{2L} - (1-\lambda) q_{2H} - c = 0 \\ 100 - q_1 - 2q_{2L} - c_L = 0 \\ 100 - q_1 - 2q_{2H} - c_H = 0 \end{cases}$$

   Solving this, we obtain

$$q_1^* = 10 - \frac{1}{5}[2c - \lambda c_L - (1-\lambda) c_H] \quad \textit{etc.}$$

   It remains to verify that $q > 0$ *and* $q_1 + q_{2L}, q_1 - q_{2H} < 100$.

4. An indivisible object is auctioned off to the highest bidder in a sealed-bid procedure. The bidder with the highest bid wins the object and pays the second highest bid. Suppose that there are two bidders, and that the bidders' valuations are statistically independent draws from the uniform distribution on the finite set of potential valuations $V = \{1, 2, ..., 99, 100\}$ (that is, probability $1/100$ for each value). Assume that both bidders know this, so this is their common prior, but each bidder i is only informed about his or her own valuation, v_i.

   (a) Formalize this as a (Bayesian) normal-form game with two players. What is a pure strategy in this game? Is it a Nash equilibrium to always bid one's valuation? Prove or disprove!

   Solution: *A pure strategy is a function* $f : V \to R$ *from one's valuation,* v_i*, to a bid,* $b_i = f(v_i)$*. The payoff to player 1, when the strategy profile*

(f, g) *is played, is*

$$\begin{aligned}\pi_1(f, g) &= \mathbb{E}[v_1 - g(v_2) \mid f(v_1) > g(v_2)] \cdot \Pr[f(v_1) > g(v_2)] \\ &\quad + \frac{1}{2}\mathbb{E}[v_1 - g(v_2) \mid f(v_1) = g(v_2)] \cdot \Pr[f(v_1) = g(v_2)]\end{aligned}$$

The strategy to always bid one's valuation is the function $f^*(v_i) \equiv v_i$. *Suppose that player* 1 *uses some strategy* f *and has valuation* v_1. *Suppose first that* $f(v_1) > v_1$. *Considering all possible bids* $b_2 \in R$ *by player 2, it is not difficult to show that 1 would never loose from instead bidding* $b_1 = v_1$ *and for some bids* b_2 *would actually gain. Likewise if* $f(v_1) < v_1$. *Since this holds for any* $v_1 \in V$, *this proves that* f^* *is a weakly dominant strategy, hence a best reply to any strategy* g *that player 2 might use, and then also* (f^*, f^*) *is a NE*

(b) Formalize this as a normal-form game with 200 $(= 2 \cdot 100)$ players. What the (pure) strategy set of a player in this game? Is it a Nash equilibrium for each player to bid his or her "type"? Prove or disprove!

Solution: *In this approach, each of the two bidders is represented by* 100 *players, one for each possible valuation* $v_i \in V$. *A pure strategy for such a player is just a bid, a real number, and it is a NE for each of the 200 players to bid his or her valuation.*

(c) In (a): Is the strategy to always bid one's valuation a weakly dominant strategy, in the sense that, in comparison with any other strategy, it never does worse and sometimes does better than that alternative strategy?

Solution: *Yes.*

5. Evolutionary stability

(a) Find all (pure or mixed) evolutionarily stable strategies in the symmetric two-player game that is obtained by first letting a fair coin decide who is the row player and who is the column player of the asymmetric game

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

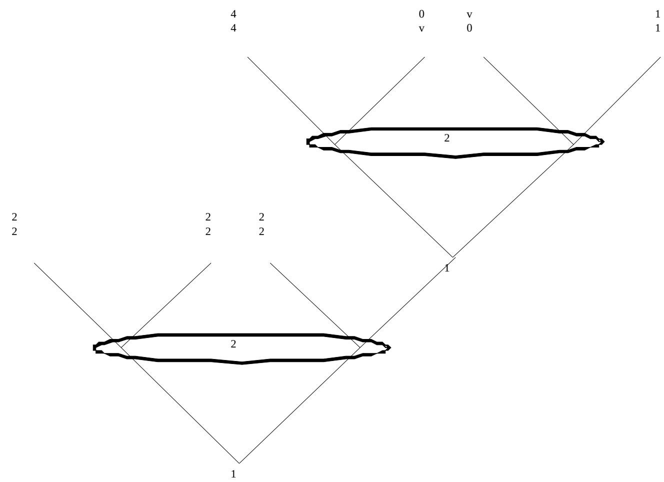
Solution: *In the symmetrized (meta-)game each player has 9 pure strategies:* TL, TC, TR, ML, MC*... The unique NE in the metagame is the strategy profile* (MC, MC), *so* MC *is the only strategy that can be an ESS. And it is an ESS, since it is its own unique best reply.*

(b) Find all (pure or mixed) evolutionarily stable strategies in

	H	D
H	$-1,-1$	$4,0$
D	$0,4$	$2,2$

Solution: *This game has only one symmetric NE, namely* (x, x) *where* $x = (2/3, 1/3)$. *This strategy is an ESS.*

(c) Write up the normal form of the extensive-form given below. How many pure strategies does each player have? Verify that it is a symmetric game. Suppose that $v = 5$. Is it an evolutionarily stable strategy to always move left? Is the strategy profile in which both players always move left a subgame-perfect equilibrium? Discuss your findings!



Solution: *Each player has 4 pure strategies. Writing up the payoff bi-matrix shows that the game is symmetric. The strategy LL is not an ESS, since it can be invaded by the "mutant" strategy RL. [RL is a best reply to LL and a better reply to itself than LL is, earning payoff 4 instead of 2.] The strategy profile* $s = (LL, LL)$ *is subgame perfect if and only if* $v \leq 4$.