SF2972 Game Theory
Problem set on extensive form games

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There are five exercises, to be handed in at the final lecture (March 10). For a bonus point, please answer all questions; at least half of your answers must be correct. Do not hesitate to contact me with questions. If you want to draw game trees, I recommend using the ps-tree commands in the PSTricks package for LaTeX .

Good luck! It was fun teaching you!

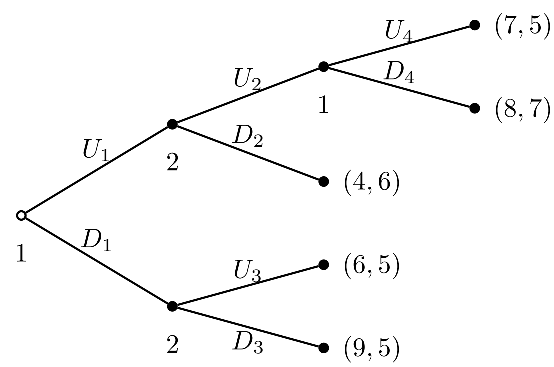
EXERCISE 1.

(a) Make exercise 103.2 from Osborne and Rubinstein.

(b) Find the subgame perfect equilibria if there is an upper bound $M \in \mathbb{N}$ on the numbers that the players can announce in the second round.

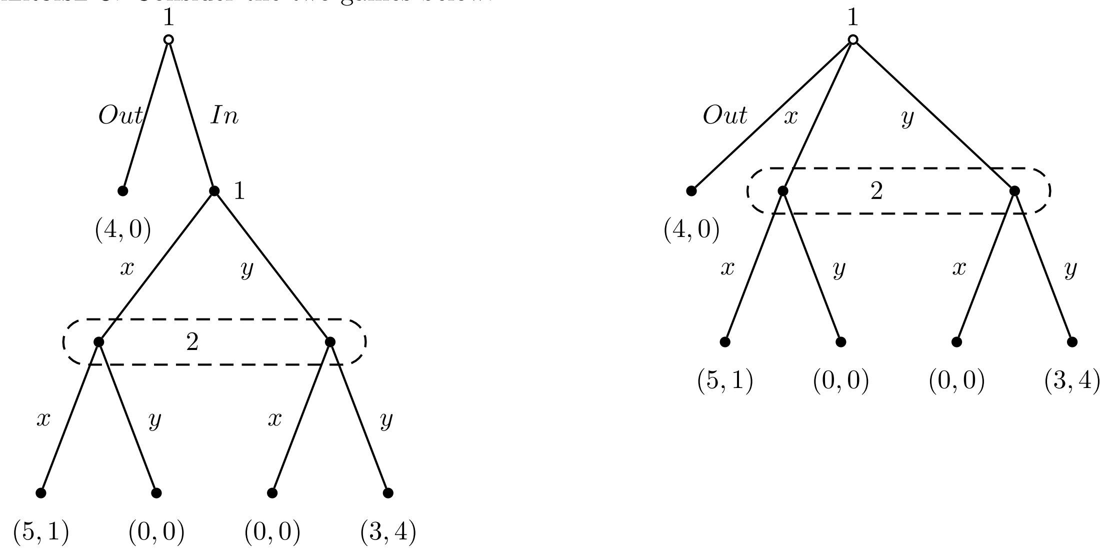
EXERCISE 2.

(a) Find all (pure) Nash and subgame perfect equilibria of the game below.



(b) Determine the corresponding strategic game. For each subgame perfect equilibrium, specify an order of iterated elimination of weakly dominated strategies that preserves the corresponding equilibrium. Can the same be done for the game's Nash equilibria?

EXERCISE 3. Consider the two games below.



(a) Do the games have the same *(i)* strategic form, *(ii)* reduced strategic form, *(iii)* agent strategic form?

(b) For both games, compute the set of sequential equilibria.

(c) Discuss the difference between the two games. What equilibrium (if any) do you find most appealing?

Exercise 4. In this exercise, we compare consistency with two other requirements on beliefs which I discussed informally during lecture 11.

A belief system μ in an extensive form game with perfect recall is ***structurally consistent***[1] if for each information set I there is a strategy profile β such that:

- I is reached with positive probability under β,
- $\mu(I)$ is derived from β using Bayes' rule.

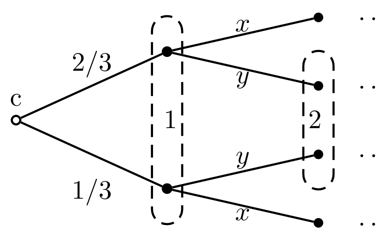
(a) In Osborne and Rubinstein, Figure 228.1, specify the set of structurally consistent belief systems.

(b) One could define a new equilibrium notion: an assessment (β, μ) that is sequentially rational and structurally consistent. Would that make sense?

An assessment (β, μ) is ***weakly consistent*** if for each information set I that is reached with positive probability under β, the beliefs $\mu(I)$ over its histories are derived from β using Bayes' rule.

Notice the difference with consistency: weak consistency imposes no constraints on beliefs over information sets that are reached with probability zero.

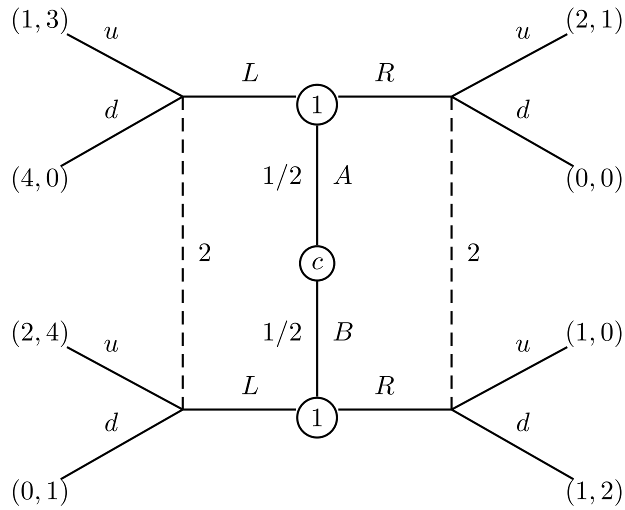
(c) Consider the information set I of player 2 in the game below. The "$\cdots$" are meant to indicate that whatever else happens in the game tree is irrelevant.



Which beliefs $\mu(I)$ are feasible under *(i)* consistency, *(ii)* structural consistency, *(iii)* weak consistency. Do you find all beliefs under weak consistency reasonable?

[1]See Osborne and Rubinstein, def. 228.1

Exercise 5. I skipped Section 12.3: it introduces unnecessary notation for special cases of extensive form games. This exercise goes through some of the essential parts.



A ***signalling game*** is played as follows (see the game above for a special case):

- ⊠ Chance starts by assigning to player 1, the ***sender***, a type. In our example, the set of types is $\{A, B\}$ and each type has equal probability.
- ⊠ Player 1 observes his type and sends a message to player 2, the ***receiver***. In our example, the set of messages is $\{L, R\}$.
- ⊠ Player 2 observes the message of player 1 (but not 1's type) and chooses an action. In our example, the set of actions is $\{u, d\}$.
- ⊠ The game ends. Payoffs can depend on types, messages, and actions.

This models a range of economic situations in which one player controls the information, but another player controls the actions. Sometimes the sender might benefit from trying to inform the receiver about his type, sometimes it is more beneficial to try to mislead the receiver.

Rather than searching for all sequential equilibria of a signalling game, it is common to make a number of simplifications: firstly, it is common to look at weakly perfect Bayesian equilibria in pure strategies (or, more precisely, behavioral strategy profiles where each player chooses in each information set a feasible action with probability one). Formally, an assessment (β, μ) is a ***weakly perfect Bayesian equilibrium*** if it is sequentially rational and weakly consistent.

Secondly, the sender may or may not be interested in sending information about his type: a weakly perfect Bayesian equilibrium in pure strategies is called a ***pooling equilibrium*** if he assigns to each type the same message and a ***separating equilibrium*** if he assigns to each type a different message. For instance, in a pooling equilibrium of our example, $\beta_1(A) = \beta_1(B)$, whereas in a separating equilibrium, $\beta_1(A) \neq \beta_1(B)$.

(a) Find all separating and pooling equilibria of the game above.

(b) Which of these equilibria are sequential equilibria?

Exercise 1a.

GAME: The game $\langle N, H, P, (\succsim_i)_{i\in N}\rangle$ has $N = \{1,2\}$,

$$H = \{\emptyset, Stop, Continue, \{(Continue, (x,y)) : x, y \in \mathbb{Z}_+\}\},$$

where $(Continue, (x,y))$ denotes the history where player 1 continues and players 1 and 2 announce nonnegative integers x and y, respectively. $P(\emptyset) = 1, P(Continue) = \{1,2\}$, and preferences $\succsim_i$ over terminal histories represented by the utility function $u_i(Stop) = 1, u_i(Continue,(x,y)) = xy$.

SUBGAME PERFECT EQUILIBRIUM: Consider the subgame after *Continue*. There is no best response to a positive integer; every nonnegative integer is a best response to 0. It follows that the unique equilibrium of the subgame is $(0,0)$ with associated payoffs $(0,0)$. Anticipating this, player 1 chooses *Stop* to reach payoff vector $(1,1)$. Conclude: the subgame perfect equilibrium has player 1 playing $(Stop, 0)$ and player 2 playing 0.

Exercise 1b. Consider the subgame after *Continue*. The unique best response to a positive integer is M; every nonnegative integer is a best response to 0. It follows that the subgame has two equilibria: $(0,0)$ with payoff vector $(0,0)$ and (M,M) with payoff vector (M^2, M^2). If player 1 anticipates equilibrium $(0,0)$, he prefers to *Stop*, if he anticipates equilibrium (M,M), he prefers to *Continue*. So in addition to the subgame perfect equilibrium in 1a, there is a new subgame perfect equilibrium where player 1 plays $(Continue, M)$ and player 2 plays M.

Exercise 2.

THE STRATEGIC GAME AND ITS EQUILIBRIA: The corresponding strategic game is

	U_2U_3	U_2D_3	D_2U_3	D_2D_3
U_1U_4	$7,5$	$7,5$	$4,6$	$4,6$
U_1D_4	$8,7^{**}$	$8,7$	$4,6$	$4,6$
D_1U_4	$6,5$	$9,5^{*}$	$6,5^{*}$	$9,5^{*}$
D_1D_4	$6,5$	$9,5^{**}$	$6,5^{*}$	$9,5^{*}$

There are seven pure Nash equilibria, marked with (one or two) stars. The subgame perfect equilibria are marked with two stars.

ITERATED ELIMINATION AND SUBGAME PERFECT EQUILIBRIA:

- ⊠ Consider the subgame perfect equilibrium (U_1D_4, U_2U_3).
- ⊠ In the subgame after history (U_1, U_2), player 1 prefers D_4 over U_4: U_1U_4 can be eliminated, as it is dominated by U_1D_4.
- ⊠ Knowing that player 1 prefers D_4 over U_4, in the subgame after history U_1, now prefers U_2 over D_2. So now U_2U_3 weakly dominates D_2U_3 and U_2D_3 weakly dominates D_2D_3.
- ⊠ In the remaining game (where pl. 1 has 3 and pl. 2 has 2 strategies left), no more actions are weakly dominated, so the elimination process stops.
- ⊠ Notice that both subgame perfect equilibria have survived the process.

ITERATED ELIMINATION AND NASH EQUILIBRIA: Nash equilibria in which player 2 chooses D_2 are always eliminated.

Exercise 3a(i). No: in the left game, player 1 has 4 pure strategies, in the right game only 3.

Exercise 3a(ii). Yes (up to an arbitrary relabeling of the strategies): in the left game, the pure strategies (Out, x) and (Out, y) of player 1 are payoff equivalent and can be replaced by a single one, Out, as in the game on the right.

Exercise 3a(iii). No: in the left game, there are 3 information sets, hence 3 players in the agent strategic form, in the right game there are 2 information sets, hence 2 players in the agent strategic form.

Exercise 3b.

Let us start with the game on the right (Uhm, why? Because it's better practice for the exam). The corresponding strategic game is

	x	y
Out	$4, 0$	$4, 0$
x	$5, 1$	$0, 0$
y	$0, 0$	$3, 4$

Since player 1's strategy y is strictly dominated by Out, sequential rationality requires that it is played with zero probability in a sequential equilibrium:

$$\beta_1(\emptyset)(y) = 0. \tag{1}$$

Distinguish two cases:

Case 1: sequential equilibria with $\beta_1(\emptyset)(x) > 0$:

- ⊠ Consistency requires that in information set $\{x, y\}$, which is reached with positive probability, the beliefs are derived from $\beta_1(\emptyset)$ using Bayes' rule. Together with (1), this gives that $\mu(\{x, y\})(x) = 1$: player 2 believes to be in node x with probability 1.
- ⊠ Player 2's unique best response to this belief is to choose x: $\beta_2(\{x, y\})(x) = 1$.
- ⊠ Consequently, player 1 prefers x, giving payoff 5, to Out, giving payoff 4: $\beta_1(\emptyset)(x) = 1$.
- ⊠ We found a candidate sequential equilibrium:

$$(\beta_1, \beta_2, \mu) = ((0, 1, 0), (1, 0), (1, 0)).$$

- ⊠ To verify consistency, notice that (β_1, β_2, μ) is the limit of the sequence of assessments

$$\left((1/n, 1 - 2/n, 1/n), (1 - 1/n, 1/n), \left(\frac{1 - 2/n}{1 - 1/n}, \frac{1/n}{1 - 1/n}\right)\right).$$

Case 2: sequential equilibria with $\beta_1(\emptyset)(x) = 0$:

- ⊠ Together with (1), this implies that $\beta_1(\emptyset)(Out) = 1$. For notational convenience, denote player 2's strategy β_2 by $(p, 1 - p)$ and the belief μ over $\{x, y\}$ by $(q, 1 - q)$.
- ⊠ Sequential rationality requires that Out is a best response of player 1. He won't play y (see (1)) and x gives expected payoff $5p$, so Out is a best response as long as $5p \leq 4$, i.e., as long as $p \in [0, 4/5]$.
- ⊠ But is every such p sequentially rational? If $0 < p \leq 4/5$, player 2 chooses both actions x and y with positive probability. Both x and y therefore have to best responses to 2's beliefs in his information set. This is true only if $q = 4(1 - q)$, i.e., if $q = 4/5$. If $p = 0$, only action y is chosen with positive probability. For this to be a best response to player 2's beliefs in his information set, we need that $q \leq 4(1 - q)$, i.e., that $q \in [0, 4/5]$.

⊠ We found the following candidates for sequential equilibria:

$$\{(\beta_1, \beta_2, \mu) = ((1,0,0), (p, 1-p), (4/5, 1/5)) \mid 0 < p \leq 4/5\},$$

and

$$\{(\beta_1, \beta_2, \mu) = ((1,0,0), (0,1), (q, 1-q)) \mid 0 \leq q \leq 4/5\}.$$

⊠ It remains to verify that these assessments are consistent. For the first class of equilibria, let $0 < p \leq 4/5$. The assessment is a limit of

$$(\beta_1^n, \beta_2^n, \mu^n) = ((1 - 1/n, 4/5n, 1/5n), (p, 1-p), (4/5, 1/5)).$$

For the second class of equilibria, let $0 \leq q \leq 4/5$. The assessment is a limit of

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 \\ q+\varepsilon \\ 1-q-\varepsilon \end{pmatrix}, (\varepsilon, 1-\varepsilon), (q+\varepsilon, 1-q-\varepsilon) \right), \qquad 0 < \varepsilon \to 0.$$

For the game on the left, a similar analysis applies, but it's a bit simpler: conditional on reaching information set $\{In\}$, sequential rationality requires the players to choose a Nash equilibrium of the 2×2 game

	x	y
x	$5,1$	$0,0$
y	$0,0$	$3,4$

This game has three Nash equilibria (in the obvious notation): $((1,0),(1,0))$ (both choose x), $((0,1),(0,1))$ (both choose y), and $((4/5,1/5),(3/8,5/8))$. Comparing the corresponding equilibrium payoffs, player 1 can choose between In and Out. We find three sequential equilibria:

1. Player 1 chooses In with probability 1, x with probability 1, player 2 chooses x with probability 1, and the belief over $\{(In, x), (In, y)\}$ assigns prob. 1 to (In, x).

2. Player 1 chooses Out with probability 1, y with probability 1, player 2 chooses y with probability 1, and the belief over $\{(In, x), (In, y)\}$ assigns prob. 1 to (In, y).

3. Player 1 chooses Out with probability 1, mixture $(4/5, 1/5)$ over (x, y), player 2 chooses mixture $(3/8, 5/8)$ over (x, y), and the belief over $\{(In, x), (In, y)\}$ assigns prob. $(4/5, 1/5)$ over histories $((In, x), (In, y))$.

Exercise 3c. The games have the same reduced strategic form: if player 1 chooses In, he knows that he immediately has to choose between x and y. In the second game, this two-step choice is modelled as a single choice.

For sequential equilibria, this matters: recall that they are trembling-hand perfect equilibria of the agent strategic form, allowing for mistakes in each information set. This makes it harder to sustain equilibria in the game on the left: there, player 1 has two information sets and consequently two opportunities for mistakes.

Which equilibrium you like most, is a matter of taste. I like the one ending in payoffs $(5,1)$ because of a forward-induction argument. If player 1 consciously gives up secure payoff 4 by not choosing Out, that must be because he's aiming to get the higher payoff 5 by choosing x. Then it makes sense for player 2 to choose x as well.

Exercise 4a. Still to do.

Exercise 4b. No, that only requires players to choose rational responses to structurally consistent beliefs. Those beliefs require only that there is *some* behavioral strategy from which it can be derived using Bayes' rule. That behavioral strategy profile need not be related to what players *actually* do.

Exercise 4c. If I is reached with positive probability under behavioral strategies β, it follows that player 1 chose y with positive probability, say $p > 0$. So the upper node is reached with probability $\frac{2}{3}p$ and the lower node with probability $\frac{1}{3}p$. By Bayes' rule, both consistent and structurally consistent beliefs must assign probability 2/3 to the upper and 1/3 to the lower node.

Weak consistency is more "liberal": if player 1 chooses x with probability 1, there are no constraints on the beliefs over player 2's information set. This seems unrealistic: *conditional* upon being in that information set, 1 must have chosen y (even if that was only by mistake or as a thought experiment), so only the beliefs $(2/3, 1/3)$ make sense.

Exercise 5. Still to do.