SF 2972 GAME THEORY
Lecture 2
Nash equilibrium

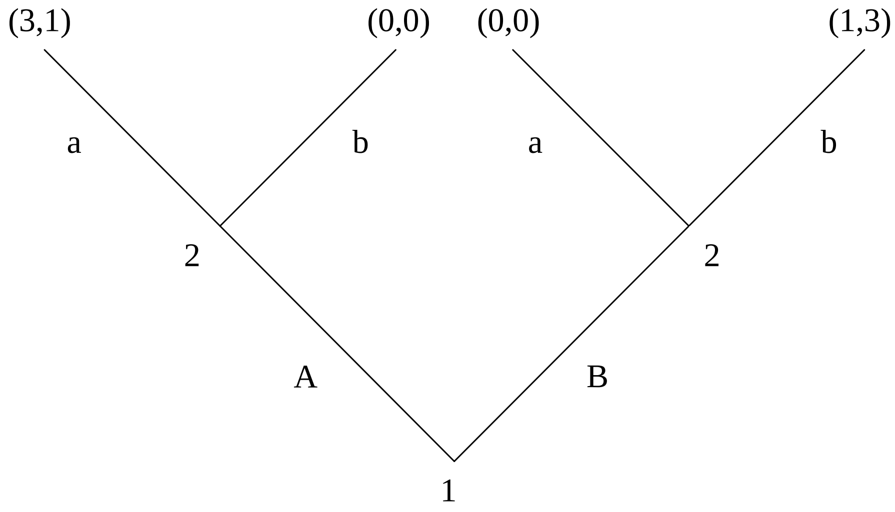
Jörgen Weibull

January 24, 2013

1 Informally about the extensive form

- Is it always better to be the more informed party?

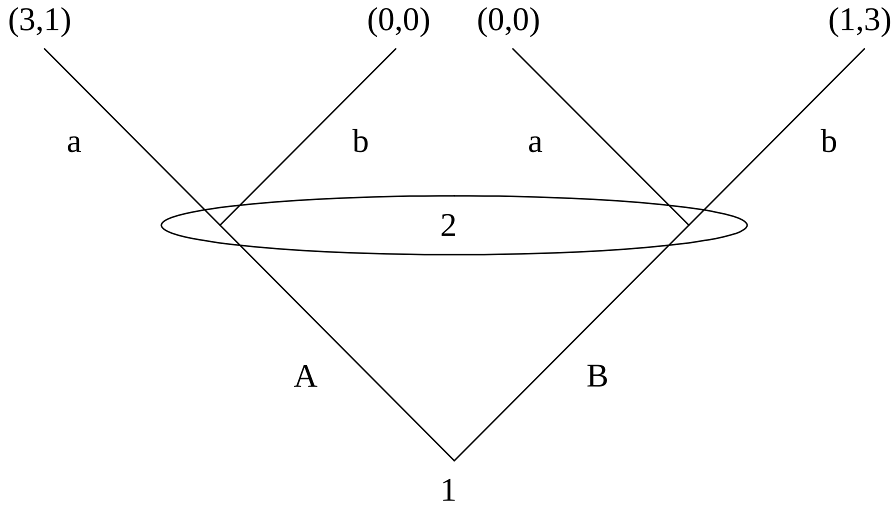
- If you are one of the players in a two-player game, would you like to be informed about the other player's move before you make your move, or would you rather prefer that the other player is informed about your move before making his?



Game 1

- Player 2 is informed about player 1's move, before 2 makes her move
- A game of perfect information
- Better to be uninformed, to be the first mover (a first-mover advantage)

- Suppose that no player is informed about the other's move:



Game 2

- In this game, player 2 cannot condition his choice on 1's action (nor can 1 condition on 2's move)

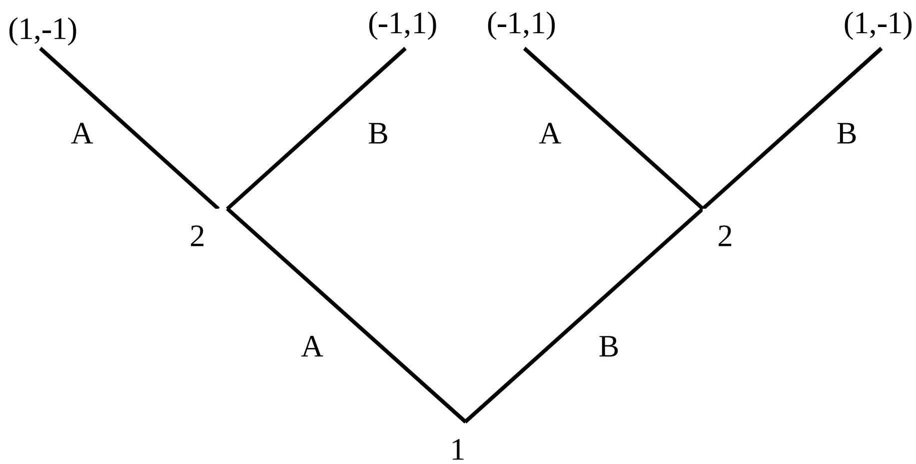
- A game of imperfect information

- Are there games in which it is better to be informed, that is, with a *second*-mover advantage?

- A fox (player 1) and a rabbit (player 2), each choosing between two locations, A and B.

- If you were the rabbit (fox), would you like to choose first or second?

- If the fox chooses first:



Game 3

2 Incomplete information

- In many strategic interactions, the actors know the "rules of the game" but not each others' preferences

- Such situations of *incomplete* information are modelled as games of *imperfect* information [Harsanyi (1967-8)]

- Create a "metagame" by introducing a neutral player, "nature", or "player 0", who makes a random draw from the set of possible games, one for each possible combination of preferences

- An extensive-form game with an initial random move by "nature".

2.1 Example

- Two competing firms who know the *prior* probabilities, μ, for the possible cost constellations, (c_L, c_L), (c_L, c_H), (c_H, c_L), (c_H, c_H)

  They use Bayes' law to infer the *posterior* probability distribution for the other firm's cost, given their own cost:

$$\text{Pr}\left[\text{Firm 2's cost is } c_L \mid \text{1's cost is } c_L\right] = \frac{\mu\left(c_L, c_L\right)}{\mu\left(c_L, c_L\right) + \mu\left(c_L, c_H\right)}$$

- Let "nature" ("player 0") first randomly draw the actual "world", ω, from four possible "worlds,"

$$\omega \in \Omega = \left\{\left(c_L, c_L\right), \left(c_L, c_H\right), \left(c_H, c_L\right), \left(c_H, c_H\right)\right\}$$

  each with a prior probability as above.

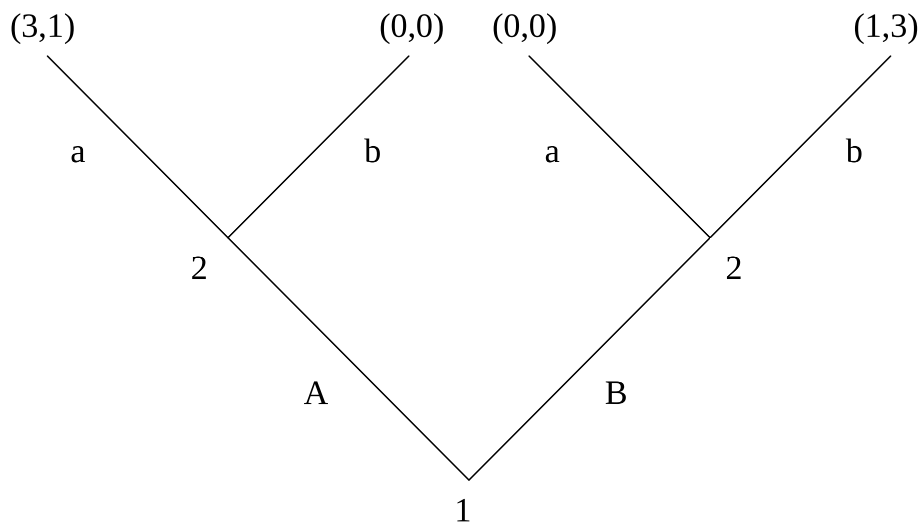
- After that initial draw by nature, inform each firm of its own cost (and only that).

3 The normal form

- The normal form is a summary representation that specifies what payoffs every player gets, for any combination of strategies.

- It is very convenient for analysis!

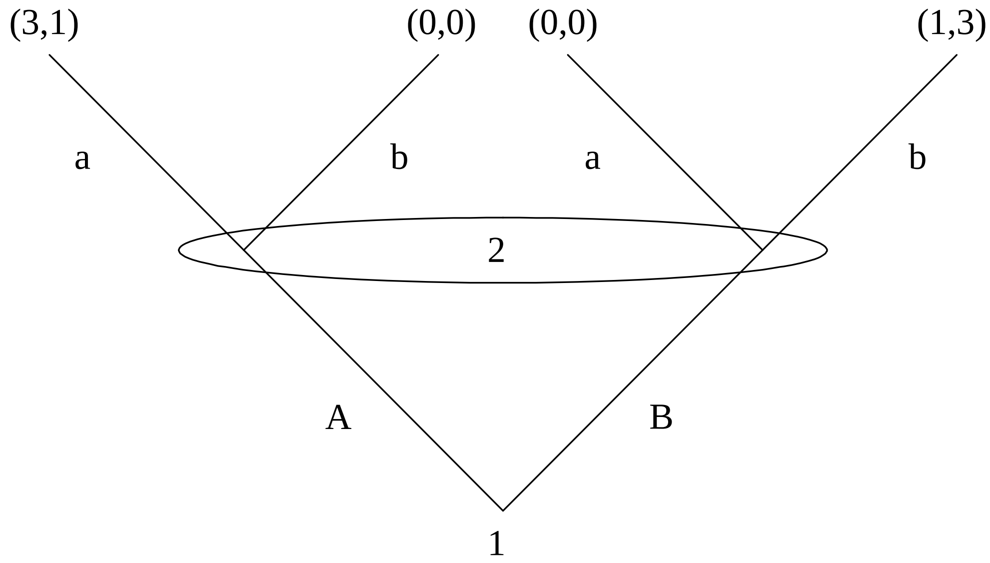
- This first part of the course will focus on normal-form analysis.



Game 1

Normal form:

	aa	ab	ba	bb
A	3, 1	3, 1	0, 0	0, 0
B	0, 0	1, 3	0, 0	1, 3



Game 2

normal form:

	a	b
A	3, 1	0, 0
B	0, 0	1, 3

- Generally:

Definition 3.1 *A* **normal-form game** *is a triplet* $G = \langle N, S, u \rangle$ *where*

(a) $N = \{1, 2, .., n\}$ *is the set of* **players**

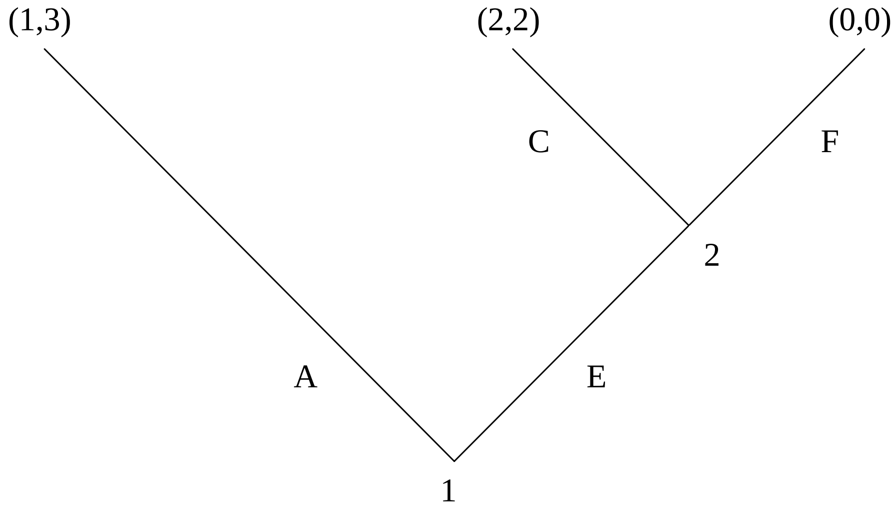
(b) $S = \times_{i \in N} S_i$ *is the set of* **strategy profiles**, $s = (s_1, .., s_n) \in S$, *and* S_i *is the strategy set of player* i

(c) $u : S \rightarrow \mathbb{R}^n$ *is the combined* **payoff function**, *where* $u_i(s) \in \mathbb{R}$ *the payoff to player* i *when profile* s *is played*

- From an abstract mathematical viewpoint, a normal-form game is any *function* $u : \times_{i \in N} S_i \rightarrow \mathbb{R}^{|N|}$

3.1 Example: Entry-deterrence

A potential entrant (player 1) into a monopolist's (player 2) market



Game 4

Normal form:

	C	F
A	1, 3	1, 3
E	2, 2	0, 0

4 Nash equilibrium a bit more formally

- Notation: Given a strategy profile $s = (s_1, .., s_n) \in S$ and any strategy $s'_i \in S_i$: write $\left(s'_i, s_{-i}\right) \in S$ for the strategy profile in which player i uses strategy s'_i while the others stick to their strategies in s.

Definition 4.1 *A strategy profile* $s \in S$ *in a normal-form game* $G = \langle N, S, u \rangle$ *is a* **Nash equilibrium** *if*

$$u_i(s) \geq u_i\left(s'_i, s_{-i}\right) \quad \forall i \in N, s'_i \in S_i$$

- Each player's strategy is a **best reply** to the others' strategies

$$\begin{cases} s_1 \in \arg\max_{s_1' \in S_1} u_1\left(s_1', s_2, .., s_n\right) \\ s_2 \in \arg\max_{s_2' \in S_2} u_2\left(s_1, s_2', .., s_n\right) \\ \dots \\ s_n \in \arg\max_{s_n' \in S_n} u_n\left(s_1, s_2, .., s_n'\right) \end{cases}$$

- Reconsider earlier examples!

5 Mathematics

5.1 Notation, concepts and theorems

1. **Useful sets**: $\mathbb{N}$ the positive integers, $\mathbb{R}$ the reals, $\mathbb{R}_+$ the non-negative reals, $\mathbb{R}_{++}$ the positive reals. Euclidean spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ etc.

2. **Standard definitions:** *open*, *closed* and *bounded* sets $X \subset \mathbb{R}^n$, the *interior* and *closure* of sets $X \subset \mathbb{R}^n$, continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ (see any math book)

Definition 5.1 *A set $X \subseteq \mathbb{R}^n$ is* **convex** *if*

$$x, y \in X \Rightarrow \lambda x + (1 - \lambda) y \in X \quad \forall \lambda \in [0, 1]$$

Definition 5.2 *A* **upper-contour set** *for a function* $f : \mathbb{R}^n \to \mathbb{R}$ *is any set* $X \subseteq \mathbb{R}^n$ *such that*

$$X = \{x \in \mathbb{R}^n : f(x) \geq \alpha\}$$

for some $\alpha \in \mathbb{R}$.

Definition 5.3 *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is quasi-concave if all its upper contour-sets are convex*

Given a function $f : X \to \mathbb{R}$, we will use the following notation for its set of maximands, points $x \in X$ at which it attains its maximal value:

$$X^* = \arg\max_{x \in X} f(x) = \{x^* \in X : f(x^*) \geq f(x) \;\; \forall x \in X\}$$

This may well be the empty set, $\varnothing$. For example, let $X = \mathbb{R}$ and $f(x) \equiv x$.

Theorem 5.1 (Weierstrass' Maxium Theorem) *If $X \subset \mathbb{R}^n$ is non-empty and compact, and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then X^* is non-empty and compact.*

Another useful result:

Proposition 5.2 *If $X \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}$ quasi-concave, then X^* is convex.*

5.2 Correspondences

A *correspondence* φ from a set X to a set Y, written $\varphi : X \rightrightarrows Y$, is a function that assigns a non-empty set $\varphi(x) \subseteq Y$ to each $x \in X$.

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ for some $n, m \in \mathbb{N}$.

Definition 5.4 $\varphi : X \rightrightarrows Y$ *is* **upper hemi-continuous (u.h.c.)** *at* $x \in X$ *if for every open set* B *containing* $\varphi(x)$ *there exists an open set* A *containing* x *such that*

$$x' \in A \cap X \;\Rightarrow\; \varphi(x') \subset B.$$

- φ is *u.h.c.* if it is u.h.c. at *each* point x in its domain X.

5.3 Berge's Maximum Theorem

(A special case.) Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be continuous, $n, k \in \mathbb{N}$, and let X be non-empty and compact. For each $y \in \mathbb{R}^k$, define

$$v(y) = \max_{x \in X} f(x, y)$$
$$\xi(y) = \arg\max_{x \in X} f(x, y)$$

Theorem 5.3 (Berge's Maximum Theorem) *If $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\xi : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is u.h.c. and $v : \mathbb{R}^k \to \mathbb{R}$ is continuous.*

5.4 Fixed-Point Theorems

Theorem 5.4 (Brouwer's Fixed-Point Theorem) *If $X \subset \mathbb{R}^n$ is non-empty, compact and convex, and $f : X \rightarrow X$ is continuous, then there exists at least one $x^* \in X$ such that $x^* = f(x^*)$.*

- A correspondence $\varphi : X \rightrightarrows Y$ is *closed-valued* (*convex-valued*) if $\varphi(x)$ is closed (convex) for every $x \in X$.

Theorem 5.5 (Kakutani's Fixed-Point Theorem) *If $X \subset \mathbb{R}^n$ is non-empty, compact and convex, and $\varphi : X \rightrightarrows X$ is convex-valued, closed-valued and u.h.c., then there exists at least one $x^* \in X$ such that $x^* \in \varphi(x^*)$.*

6 Existence of Nash equilibrium

On the basis of the above mathematics, one can establish existence of Nash equilibrium for a wide class of games.

By a **Euclidean game** we mean a game in which each player's strategy set is a subset of some Euclidean space. [For each player $i \in N$: $S_i \subseteq \mathbb{R}^{m_i}$ for some $m_i \in \mathbb{N}$.]

Theorem 6.1 (T1) *Let $G = \langle N, S, u \rangle$ be a Euclidean game in which each strategy set S_i is non-empty, compact and convex, and each payoff function $u_i : S \to \mathbb{R}$ is continuous. If each payoff function u_i is quasi-concave in $s_i \in S_i$ (for any given $s_{-i} \in S_{-i}$) then G has at least one Nash equilibrium.*

Proof: For each player i, let $\beta_i : S \rightrightarrows S_i$ be the player's best-reply correspondence, defined by

$$\beta_i(s) = \arg\max_{s'_i \in S_i} u_i\left(s'_i, s_{-i}\right)$$

By Weierstrass' Maximum Theorem, $\beta_i(s)$ is non-empty and compact for every $s \in S$. By quasi-concavity and the convexity of S_i, $\beta_i(s)$ is convex. Hence also the combined best-reply correspondence $\beta : S \rightrightarrows S$, defined by $\beta(s) = \times_{i \in I} \beta_i(s)$, has these properties. By Berge's Maximum Theorem, each correspondence β_i is upper hemi-continuous, and hence so is β. Thus all conditions in Kakutani's Fixed-Point Theorem are met, so β admits a fixed point. A strategy profile $s \in S$ is a fixed point under β if and only if s is a Nash equilibrium. **Q.E.D.**