SF 2972 GAME THEORY
Lecture 3
Finite games in normal form, part I

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- Let $G = \langle N, S, \pi \rangle$ be a finite game, where
  - N is the finite set of (personal) **players**
  - $S = \times_{i \in N} S_i$ is the finite set of **strategy profiles** $s = (s_1, .., s_n)$
  - π is the joint **payoff function**, $\pi_i (s_1, .., s_n) \in \mathbb{R}$ being the payoff to player i when profile s is played

- We will henceforth consider the **mixed-strategy extension** $\tilde{G} = \langle N, \boxdot (S), \tilde{\pi} \rangle$ of G, the normal-form game in which a strategy for each player i is a probability distribution over the finite set S_i

- We need to specify $\boxdot (S)$ and $\tilde{\pi} : \boxdot (S) \to \mathbb{R}^n$

1 Mixed-strategy sets

Let m_i be the number of pure strategies available to player i: $m_i = |S_i|$

- The **mixed-strategy set** for player $i \in N$ is the **unit simplex** spanned by his/her pure strategies:

$$\Delta_i = \Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{h=1}^{m_i} x_{ih} = 1\}$$

- The **support** of any given mixed strategy x_i: *supp*$(x_i) = \{h \in S_i : x_{ih} > 0\}$

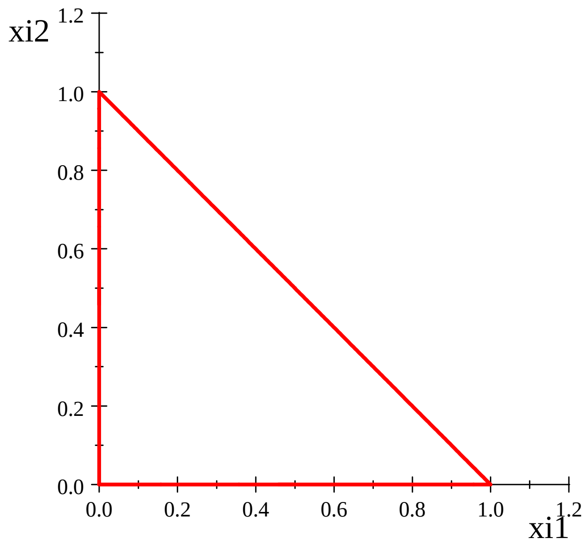
- The **vertices** of Δ_i are the **unit vectors**, e_i^h for $i \in N$, $h \in S_i$ [interpreted as pure strategies]

- **Interior** or **completely mixed** strategies:

$$int(\Delta_i) = \{x_i \in \Delta_i : x_{ih} > 0 \; \forall h \in S_i\}$$

then all i's pure strategies are played with positive probability

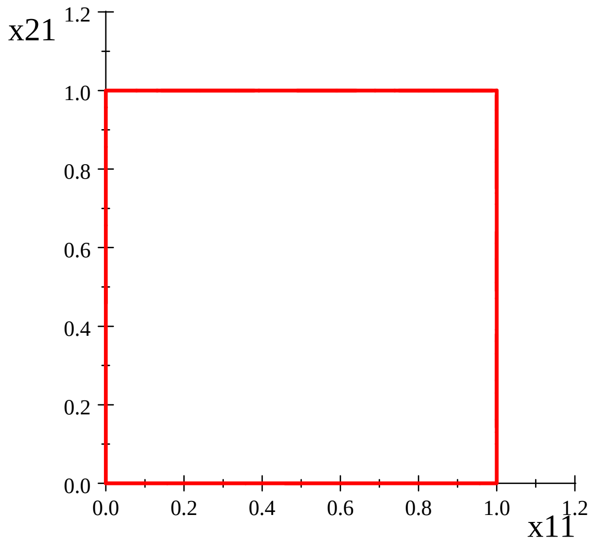
Example: $m_i = 3$



- The **polyhedron** of mixed-strategy **profiles**:

$$X = \boxdot(S) = \times_{i \in N} \Delta_i = \times_{i \in N} \Delta(S_i)$$

- Example: $\boxdot(S)$ when $n = m_1 = m_2 = 2$:



- Draw a picture of $\boxdot(S)$ when $n = m_2 = 2$ and $m_1 = 3$

For any player $i \in N$ and pure strategy $s_i = h \in S_i$, write $x_i(s_i)$ for x_{ih}

- The payoff function $\tilde{\pi}_i : \boxdot(S) \rightarrow \mathbb{R}$ of each player $i \in N$ assigns to each mixed-strategy profile $x = (x_1, .., x_n) \in \boxdot(S)$ the associated *expected value* of i's payoff when strategy profile x is played:

$$\tilde{\pi}_i(x) = \sum_{s \in S} \left[\Pi_{j \in I} x_j\left(s_j\right)\right] \pi_i(s)$$

- Note the assumed statistical independence between different players' randomizations

Example 1.1 *The previously studied partnership game,*

$$\begin{array}{ccc} & C & F \\ C & 3,3 & -1,4 \\ F & 4,-1 & -2,-2 \end{array}$$

Here the payoff matrix to player 1 is

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

and that to player 2 is $B = A^T$ *(such games are called symmetric). We thus have*

$$\tilde{\pi}_1(x) = x_1 \cdot Ax_2 = 3 \cdot x_{11}x_{21} - 1 \cdot x_{11}x_{22} + 4 \cdot x_{12}x_{21} - 2 \cdot x_{12}x_{22}$$

2 Dominance relations

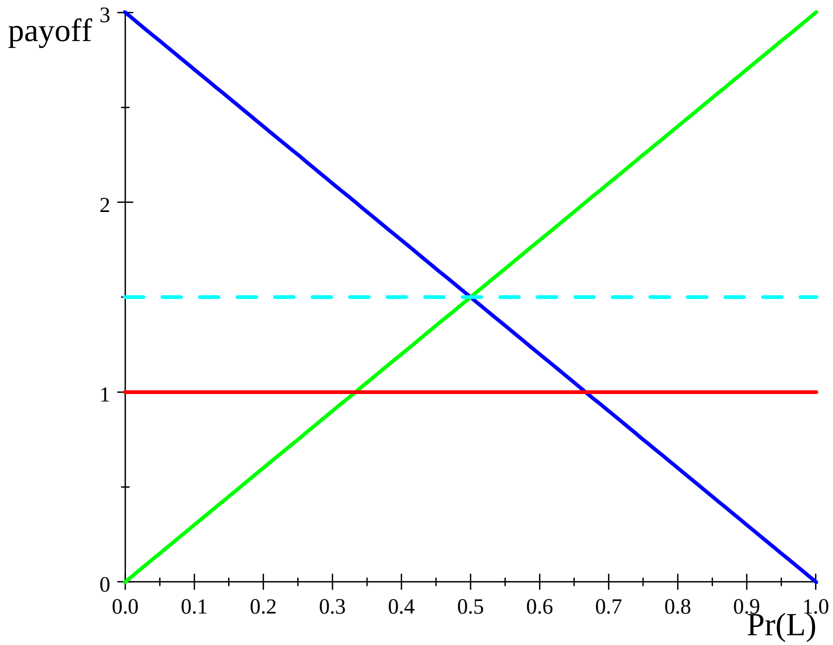
Definition 2.1 $x_i^* \in \Delta_i$ **strictly dominates** $x_i' \in \Delta_i$ *if* $\tilde{\pi}_i(x_i^*, x_{-i}) > \tilde{\pi}_i(x_i', x_{-i})$ *for all* $x \in \boxdot(S)$.

Definition 2.2 $x_i^* \in \Delta_i$ **weakly dominates** $x_i' \in \Delta_i$ *if* $\tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i})$ *for all* $x \in \boxdot(S)$ *with* $>$ *for some* $x \in \boxdot(S)$.

Definition 2.3 $x_i^* \in \Delta_i$ *is* **weakly dominant** *if it weakly dominates all strategies* $x_i' \in \Delta_i$. *A strategy that is not weakly dominated is called* **undominated**. *A strategy that strictly dominates all other strategies is* **strictly dominant**.

- Example: payoff matrix to player 1

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}$$



payoff
3
2
1
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Pr(L)

- Iterated elimination of strictly dominated pure strategies:

$$G = \begin{bmatrix} 3,3 & 1,0 & 6,1 \\ 0,1 & 0,0 & 4,2 \\ 1,6 & 2,4 & 5,5 \end{bmatrix}$$

- A game is called **dominance solvable** if the iterated elimination of strictly dominated pure strategies results in a **single** pure-strategy profile.

3 Best replies

- The i:th player's **pure**-*strategy* **best-reply correspondence** $\beta_i : \boxdot(S) \rightrightarrows S_i$ is defined by

$$\beta_i(x) = \{h \in S_i : \tilde{\pi}_i(e_i^h, x_{-i}) \geq \tilde{\pi}_i(e_i^k, x_{-i}) \ \forall k \in S_i\}$$

- Mixed strategies cannot give higher payoffs than pure:

$$\beta_i(x) = \{h \in S_i : \tilde{\pi}_i(e_i^h, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i}) \ \forall x_i' \in \boldsymbol{\Delta}_i\}.$$

- The i:th player's **mixed**-*strategy* **best-reply correspondence** $\tilde{\beta}_i$: $\boxdot(S) \rightrightarrows \boldsymbol{\Delta}_i$ is defined by:

$$\begin{aligned} \tilde{\beta}_i(x) &= \{x_i^* \in \boldsymbol{\Delta}_i : \tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i}) \ \forall x_i' \in \boldsymbol{\Delta}_i\} \\ &= \{x_i^* \in \boldsymbol{\Delta}_i : \textit{supp}(x_i^*) \subset \beta_i(x)\} \end{aligned}$$

- Note that $\tilde{\beta}_i(x)$ is a (non-empty) subsimplex

- The **combined pure** *BR-correspondence* $\beta : \boxdot(S) \rightrightarrows S$:

$$\beta(x) = \times_{i \in N} \beta_i(x)$$

- The **combined mixed** *BR-correspondence* $\tilde{\beta} : \boxdot(S) \rightrightarrows \boxdot(S)$:

$$\tilde{\beta}(x) = \times_{i \in N} \tilde{\beta}_i(x)$$

3.1 Dominance vs. best replies

- Pure best replies are not strictly dominated

- If a pure strategy is not strictly dominated, is it then a best reply to some belief?

- Pure best replies to *interior* strategy profiles are undominated

- If a pure strategy is undominated, is it then a best reply to some interior belief?

Proposition 3.1 (Pearce, 1984) *Suppose* $n = 2$*. Then* $s_i \in S_i$ *is not strictly dominated iff* $s_i \in \beta_i(x)$ *for some* $x \in \boxdot(S)$*, and* $s_i \in S_i$ *is undominated iff* $s_i \in \beta_i(x)$ *for some* $x \in int(\boxdot(S))$*.*

4 Rationalizability

- Consider a finite game in normal form, $G = \langle N, S, \pi \rangle$ and assume

  A1 (*Rationality)*: Each player i forms a probabilistic belief $\mu_{ij} \in \Delta\left(S_j\right)$ about every other player j's strategy choice, a belief that does not contradict any information or knowledge that player i has, and player i chooses a (pure or mixed) strategy that maximize his or her expected payoff, assuming statistical independence between other player's strategy choices.

  A2 (*Common Knowledge*): The game G and the players' rationality (A1) is common knowledge among the players: each player knows G and that (A1) holds for all players, knows that all players know this, and knows that all players know that all players know this etc. *ad infinitum*.

- **Question**: What is the logical implication of A1 and A2?

- **Answer**: rationalizability!

1. For any $X_j \subset \Delta\left(S_j\right)$, let $X = \times_{j=1}^{n} X_j$ and write

$$\tilde{\beta}_i(X) = \left\{x_i^* \in \Delta(S_i) : x_i^* \in \tilde{\beta}_i(x) \text{ for some } x \in X\right\}$$

2. Write $B_j(0) = \Delta\left(S_j\right)$ and $B(0) = \times_{j=1}^{n} B_j(0)$. [Thus $B(0) = \boxdot(S)$]

3. Define the set sequence $\langle B(t)\rangle_{t\in\mathbb{N}}$ recursively by

$$B_i(t+1) = \tilde{\beta}_i[C(t)]$$

where $B(t) = \times_{j=1}^{n} B_j(t)$, $C(t) = \times_{j=1}^{n} C_j(t)$ and

$C_j(t)$ is the *convex hull* of $B_j(t)$

4. Note that $B_i(t+1) \subseteq B_i(t)$ for all t and i.

Definition 4.1 (Pearce, 1984) *A strategy $x_i \in \Delta(S_i)$ is* **rationalizable for player** i *if $x_i \in B_i$, where*

$$B_i = \cap_{t \in \mathbb{N}} B_i(t).$$

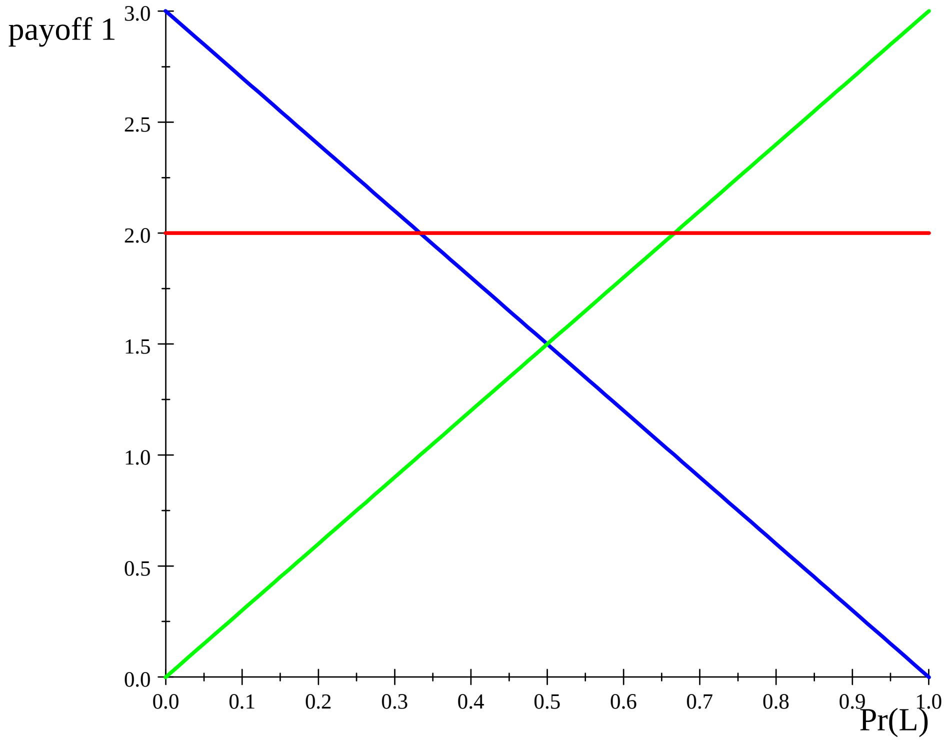
- Let C_i be the convex hull of B_i

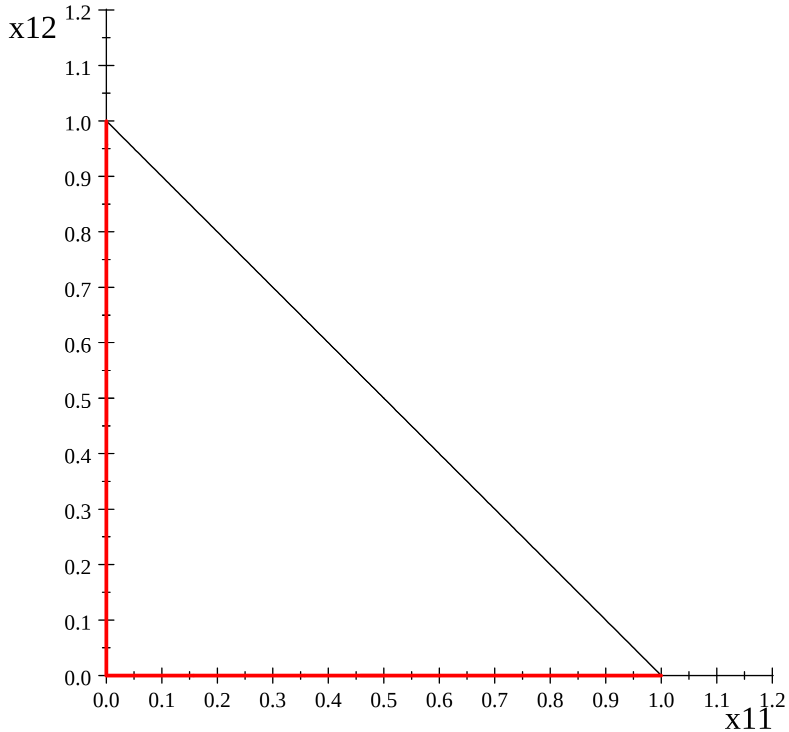
Proposition 4.1 *For each i: $B_i \neq \varnothing$ and $C_i = \Delta(T_i)$ for some non-empty subset $T_i \subset S_i$*

- A set $B_i(t)$ is not necessarily convex:

Example 4.1 *Consider player 1 with payoff matrix*

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$$





$$B_1 = \{x_1 \in \Delta_1 : x_{11}x_{12} = 0\} \quad \neq \quad C_1 = \Delta_1$$

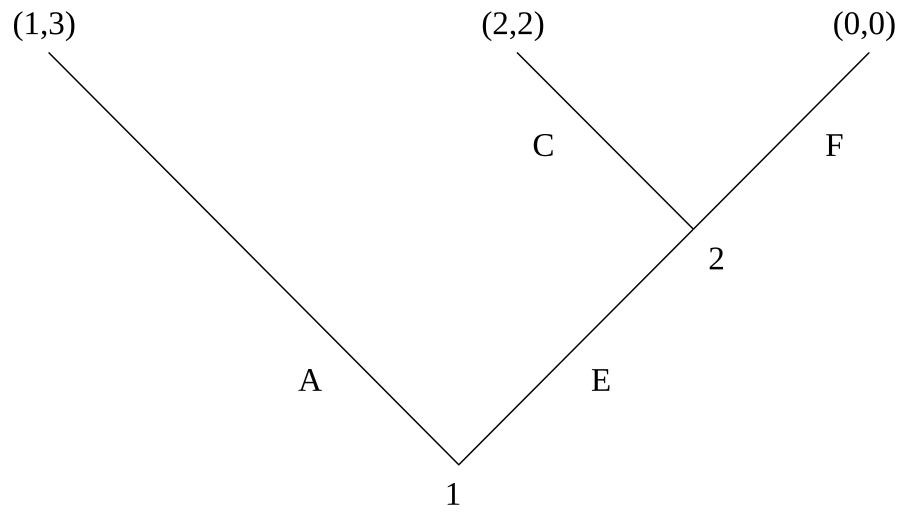
5 Nash equilibrium

Definition 5.1 $X^{NE} = \left\{ x \in \boxdot(S) : x \in \tilde{\beta}(x) \right\}$.

Definition 5.2 $x \in X^{NE}$ *is strict if* $\tilde{\beta}(x) = \{x\}$.

- A NE strategy cannot be *strictly* dominated, but may be *weakly* dominated. Example?

Example 5.1 *Reconsider the entry-deterrence game:*



Game 4

The strategy profile $s = (A, F)$ *is a Nash equilibrium! But* F *is weakly dominated by* C. *(The game has infinitely dominated Nash equilibria. Find them!)*

	C	F
A	1, 3	1, 3
E	2, 2	0, 0

5.1 Existence

Theorem 5.1 (Nash, 1950) $\boxdot^{NE} \neq \varnothing$.

Two alternative proofs:

1. Application of Kakutani's fixed-point theorem (Nash's first proof)

2. Application of Brouwer's fixed-point theorem (Nash's second proof). This inspired Arrow's and Debreu's proof of the existence of Walrasian equilibrium in general-equilibrium theory.

Proof 1: The polyhedron $\boxdot(S)$ is non-empty, convex and compact. Berge's Maximum Theorem implies that $\tilde{\beta} : \boxdot \rightrightarrows \boxdot$ is upper hemi-continuous. We saw that $\tilde{\beta}(x)$ is a non-empty convex and closed set. Hence, Kakutani's Fixed-Point Theorem applies, so $x^* \in \tilde{\beta}(x^*)$ for at least one $x^* \in \boxdot$.

Proof 2: Let $\pi^+_{ih}(x) = \max\left\{0, \tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x)\right\}$ and define $f : \boxdot(S) \to \boxdot(S)$ by

$$f_{ih}(x) = \frac{x_{ih} + \pi^+_{ih}(x)}{1 + \sum_{k \in S_i} \pi^+_{ik}(x)} \qquad \forall i \in N, h \in S_i$$

Clearly f is continuous and thus has a fixed point by Brouwer's Fixed-Point Theorem. Not difficult to verify that each fixed point $x^* \in X^{NE}$.