SF 2972 GAME THEORY

Lecture 4

Finite games in normal form, part II

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1 Examples of "implausible" Nash equilibria

Example 1.1 *The (infinitely many) dominated Nash equilibria in the entry-deterrence game.*

Example 1.2 *A firm offers a wage* $w \in W \subset [0, 100]$ *to a job candidate, who can accept or reject the offer. If accept,* $y = 1$*, the firm's profit is* $\pi = 100 - w$ *and the job candidate's utility is* $v(w)$*, where* $v : \mathbb{R}_+ \to \mathbb{R}$ *is an increasing function. If the candidate rejects the offer,* $y = 0$*, the firm's profit is zero and the candidate's utility is* $\bar{v} = v(w_0)$*, for some* $w_0 \in W$ *(the candidate's outside option). Viewed as a two-player game, in which the firm strives to maximize its profit and the job candidate his or her utility, we have* $G = \langle N, S, u \rangle$*, where* $N = \{1, 2\}$ *and*

$S_1 = W$*, the* **strategy set** *of player 1, the firm*

$S_2 = F = \{0, 1\}^W$*, the set of functions* $f : W \to \{0, 1\}$*. This is the* **strategy set** *of player 2, the candidate*

The firm's **payoff function***:* $u_1(w, f) = (100 - w) \cdot f(w) \quad \forall w \in W, f \in F$

The candidate's **payoff function***:* $u_2(w, f) = u(w) \cdot f(w) + u\left(w^0\right) \cdot [1 - f(w)]$

Q1: What are the Nash equilibria if $W = \{0, 1, ..., 99, 100\}$ *and* $w_0 = 30$*?*

Q2: What are the Nash equilibria if $W = [0, 100]$ *and* $w_0 = 30$*?*

- Can one, in finite normal-form games, discard such implausible Nash equilibria by first principles?

- We will study two such refinements:

  perfection (Selten, 1975) and

  properness (Myerson, 1978)

2 Perfect equilibria

- The most well-known refinement of Nash equilibrium: *"trembling hand" perfection* Selten (1975)

- Discards NE which are not robust to small "trembles" in the players' strategy choices.

- Selten argues that rationality should be viewed as the limit of bounded rationality as the bounds vanish

- Imagine that players sometimes (maybe extremely rarely) make mistakes, and are aware of this risk, for themselves and others

- Recall that a strategy profile x is a NE iff

$$h \notin \tilde{\beta}_i(x) \;\Rightarrow\; x_{ih} = 0$$

Definition 2.1 *Given any $\varepsilon \in (0,1)$, a strategy profile x is ε-***perfect** *if it is interior ($x_{ih} > 0 \; \forall i \in I, h \in S_i$) and such that*

$$h \notin \tilde{\beta}_i(x) \quad \Rightarrow \; x_{ih} \leq \varepsilon$$

- All pure strategies have positive probability of being used (if only by mistake), but no suboptimal strategy is used with probability above the given $\varepsilon > 0$

- For $\varepsilon > 0$ small: as if play is "almost rational"

- The following definition is equivalent to the original definition in Selten (1975):

Definition 2.2 *A strategy profile* $x^* \in \boxdot(S)$ *is* **perfect** *if it is the limit of some sequence of* ε*-perfect strategy profiles* x^ε*, where* $\varepsilon \to 0$*.*

Write $X^{PE} \subseteq \boxdot(S)$ for the set of perfect equilibria (in a given game). It follows that:

1. Perfect strategy-profiles are Nash equilibria: $X^{PE} \subset X^{NE}$

2. All strict Nash equilibria are perfect

3. All completely mixed Nash equilibria are perfect

- Perfection rules out the implausible equilibria in the entry-deterrence game:

	C	F
A	1, 3	1, 3
E	2, 2	0, 0

  - Calculations in class
  - $\Rightarrow$ unique perfect equilibrium, (E, C)

- Perfection also rules out the implausible equilibria in all finite versions of the firm-worker example:
  - Suppose that $W = \{0, 1, .., 100\}$ and $w^0 = 30$.
  - Calculations in class
  - $\Rightarrow \exists$ two perfect equilibrium wages: $W^{PE} = \{30, 31\}$.

- One can show that
  - Every finite game has at least one perfect equilibrium
  - Perfection rules out weakly dominated strategies

Proposition 2.1 $X^{PE} \neq \varnothing$. *Every perfect equilibrium is an undominated Nash equilibrium. The converse is true in two-player games.*

- Myerson (1978) pointed out that perfection is not robust to addition of strictly dominated strategies.

Example 2.1 *Add a "dumb" strategy in the entry-deterrence game (the potential entrant may shoot himself in the foot):*

	C	F
A	$1,3$	$1,3$
E	$2,2$	$0,0$
D	$-4,0$	$-4,1$

In this game, (A, F) is perfect! [Since F is a better reply than C against 1's strategy D.]

Calculations in class

3 Proper equilibria

- Myerson (1978): Players have "trembling hands"- but are better at avoiding more costly mistakes than less costly mistakes

- This solution concept requires robustness against trembles that are *an order of magnitude* less likely to more costly mistakes

- In the example above, with the added "dumb" strategy, player 2 should arguably put (much) less probability on 1 playing D than E

- While (A, F) is perfect, is it then not "proper"?

- How formalize this notion?

Definition 3.1 (Myerson, 1978) *Given $\varepsilon > 0$, a strategy profile x is ε-* **proper** *if it is interior ($x_{ih} > 0$ $\forall i \in I, h \in S_i$) and such that*

$$\tilde{\pi}_i\left(e_i^h, x_{-i}\right) < \tilde{\pi}_i\left(e_i^k, x_{-i}\right) \quad \Rightarrow \quad x_{ih} \leq \varepsilon \cdot x_{ik}$$

Definition 3.2 (Myerson, 1978) *$x^* \in \boxdot(S)$ is* **proper** *if it is the limit of some sequence of ε-proper strategy profiles x^ε, where $\varepsilon \to 0$.*

- Let $X^{PR} \subseteq \boxdot(S)$ denote the set of proper equilibria.

- Note that any completely mixed Nash equilibrium is ε-proper for all $\varepsilon > 0$ and hence:

$$X^{NE} \cap int(\boxdot) \subset X^{PR}$$

- It is not difficult to show that any strict equilibrium is proper.

- Moreover, the mixed-strategy extension of any finite game has at least one proper equilibrium, and all proper equilibria are perfect:

Proposition 3.1 (Myerson, 1978) $\varnothing \neq X^{PR} \subseteq X^{PE}$.

Example 3.1 *Reconsider the augmented entry-deterrence game*

	C	F
A	$1, 3$	$1, 3$
E	$2, 2$	$0, 0$
D	$-4, 0$	$-4, 1$

(A, F) *is not proper.*

Informally: the mistake D is more costly to player 1 than the mistake E, when play is close to (A, F)*, and thus 2 guards herself more against 1's deviation to E than to 1's deviation to D, which leads 2 to play C, not D.*

Formally: for any $\varepsilon > 0$ *and* ε*-proper strategy profile* x^{ε}*:* $x_{13}^{\varepsilon} \leq \varepsilon \cdot x_{12}^{\varepsilon}$ *and thus*

$$\tilde{\pi}_2 \left(e_1^2, x_1^{\varepsilon}\right) - \tilde{\pi}_2 \left(e_2^2, x_1^{\varepsilon}\right) = 2 \cdot x_{12}^{\varepsilon} - 1 \cdot x_{13}^{\varepsilon} \geq (2 - \varepsilon)\, x_{12}^{\varepsilon} > 0$$

- Properness has a powerful implication for extensive-form analysis: Every proper equilibrium, in any given normal-form game G, induces a *sequential equilibrium* in every *extensive-form game* with the normal form G

- The next 4 lectures, taught by Mark Voorneveld, will introduce you to extensive-form analysis and sequential equilibrium.

Summary for finite normal-form games:

$$\varnothing \neq X^{PR} \subseteq X^{PE} \subseteq X^{UDNE} \subseteq X^{NE} \subseteq X^{RAT}$$

THE END