SF2972: Game theory

The 2012 ‘Nobel prize in economics’:
awarded to Alvin E. Roth and Lloyd S. Shapley for “the theory of stable allocations and the practice of market design”

The related branch of game theory is often referred to as **matching theory**, which studies the design and performance of platforms for transactions between agents. Roughly speaking, it studies who interacts with whom, and how: which applicant gets which job, which students go to which universities, which donors give organs to which patients, and so on.

Plan

Many methods for finding desirable allocations in matching problems are variants of two algorithms:

1. The top trading cycle algorithm
2. The deferred acceptance algorithm

For each of the two algorithms, I will do the following:

- State the algorithm.
- State nice properties of outcomes generated by the algorithm.
- Solve an example using the algorithm.
- Describe application(s).
- Give you a homework exercise.

The top trading cycle (TTC) algorithm: reference

- L.S. Shapley and H. Scarf, 1974, On Cores and Indivisibility. *Journal of Mathematical Economics* 1, 23–37.
- The algorithm is described in section 6, p. 30, and attributed to David Gale.

The top trading cycle (TTC) algorithm: statement

Input: Each of $n \in \mathbb{N}$ agents owns an indivisible good (a house) and has strict preferences over all houses.
Convention: agent i initially owns house h_i.
Question: Can the agents benefit from swapping houses?
TTC algorithm:

1. Each agent i points to her most preferred house (possibly i's own); each house points back to its owner.
2. This creates a directed graph. In this graph, identify cycles.
   - Finite: cycle exists.
   - Strict preferences: each agent is in at most one cycle.
3. Give each agent in a cycle the house she points at and remove her from the market with her assigned house.
4. If unmatched agents/houses remain, iterate.

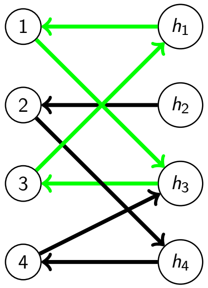
The top trading cycle (TTC) algorithm: nice properties

1. The TTC assignment is such that no subset of owners can make all of its members better off by exchanging the houses they initially own in a different way.
   - In technical lingo: the TTC outcome is a core allocation.
2. The TTC assignment is the only such assignment.
   - Unique core allocation.
3. It is never advantageous to an agent to lie about preferences if the TTC algorithm is used.
   - The TTC algorithm is strategy-proof.

The top trading cycle (TTC) algorithm: example

Agents' ranking from best (left) to worst (right):

1 : (h_3, h_2, h_4, h_1)
2 : (h_4, h_1, h_2, h_3)
3 : (h_1, h_4, h_3, h_2)
4 : (h_3, h_2, h_1, h_4)

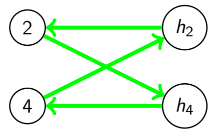


- Cycle: $(1, h_3, 3, h_1, 1)$.
- So: 1 get h_3 and 3 gets h_1. Remove them and iterate.

The top trading cycle (TTC) algorithm: example

Only agents 2 and 4 left with updated preferences:

2 : (h_4, h_2)
4 : (h_2, h_4)



- Cycle: $(2, h_4, 4, h_2, 2)$.
- So: 2 gets h_4 and 4 gets h_2. Done!
- Final match:

$$(1, h_3), (2, h_4), (3, h_1), (4, h_2).$$

The top trading cycle (TTC) algorithm: application 1

- A. Abdulkadiroğlu and T. Sönmez, 2003. School Choice: A Mechanism Design Approach. *American Economic Review* 93, 729–747.
- How to assign children to schools subject to priorities for siblings and distance?

Input:

- Students submit strict preferences over schools
- Schools submit strict preferences over students based on priority criteria and (if necessary) a random number generator

Modified TTC algorithm:

1. Each remaining student points at her most preferred unfilled school; each unfilled school points at its most preferred remaining student.
2. Cycles are identified and students in cycles are matched to the school they point at.
3. Remove assigned students and full schools.
4. If unmatched students remain, iterate.

The top trading cycle (TTC) algorithm: application 2

- A.E. Roth, T. Sönmez, M.U. Ünver, 2004. Kidney Exchange. *Quarterly Journal of Economics* 119, 457–488.
- A case with patient-donor pairs: a patient in need of a kidney and a donor (family, friend) who is willing to donate one.
- Complications arise due to incompatibility (blood/tissue) groups, etc.
- So look at trading cycles: patient 1 might get the kidney of donor 2, if patient 1 gets the kidney of donor 1, etc.

The top trading cycle (TTC) algorithm: homework exercise 6

Apply the TTC algorithm to the following case:

1 : $(h_5, h_2, h_1, h_3, h_4)$
2 : $(h_5, h_4, h_3, h_1, h_2)$
3 : $(h_4, h_2, h_3, h_5, h_1)$
4 : $(h_2, h_1, h_5, h_3, h_4)$
5 : $(h_2, h_4, h_1, h_5, h_3)$

The deferred acceptance (DA) algorithm: reference

- D. Gale and L.S. Shapley, 1962, College Admissions and the Stability of Marriage. *American Mathematical Monthly* 69, 9–15.
- Only seven pages...
- ...and, yes, stability of marriage!

The deferred acceptance (DA) algorithm: marriage problem

- Men and women have strict preferences over partners of the opposite sex
  - You may prefer staying single to marrying a certain partner
- A *match* is a set of pairs of the form (m, w), (m, m), or (w, w) such that each person has exactly one partner.
- Person i is *unmatched* if the match includes (i, i).
- i is *acceptable* to j if j prefers i to being unmatched.
- Given a proposed match, a pair (m, w) is *blocking* if both prefer each other to the person they're matched with.
  - m prefers w to his match-partner
  - w prefers m to her match-partner
- A match is *unstable* if someone has an unacceptable partner or if there is a blocking pair. Otherwise, it is *stable*.
- A match is *man-optimal* if it is stable and there is no other stable match that some man prefers. Woman-optimal analogously.

The deferred acceptance (DA) algorithm: statement

Input: A nonempty, finite set M of men and W of women. Each man (woman) ranks acceptable women (men) from best to worst.

DA algorithm, men proposing:

1. Each man proposes to the highest ranked woman on his list.
2. Women hold at most one offer (her most preferred acceptable proposer), rejecting all others.
3. Each rejected man removes the rejecting woman from his list.
4. If there are no new rejections, stop. Otherwise, iterate.
5. After stopping, implement proposals that have not been rejected.

Remarks:

1. DA algorithm, women proposing: switch roles!
2. Deferred acceptance: receiving side defers final acceptance of proposals until the very end.

The deferred acceptance (DA) algorithm: nice properties

1. The algorithm ends with a stable match.
   - By construction, no person is matched to an unacceptable candidate.
   - No (m, w) can be a blocking pair: if m strictly prefers w to his current match, he must have proposed to her and been rejected in favor of a candidate that w liked better. That is, w finds her match better than m.
2. This match is man-optimal (woman-pessimal).
3. Men have no incentives to lie about their preferences, women might.
   - Strategy-proof for men
   - See homework exercise
4. There is no mechanism that always ends in a stable match and that is strategy-proof for all participants.

The deferred acceptance (DA) algorithm: example

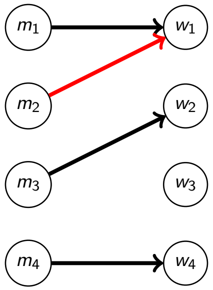
- For convenience $|M| = |W| = 4$.
- All partners of opposite sex are acceptable.
- Ranking matrix:

	w_1	w_2	w_3	w_4
m_1	$1, 3$	$2, 3$	$3, 2$	$4, 3$
m_2	$1, 4$	$4, 1$	$3, 3$	$2, 2$
m_3	$2, 2$	$1, 4$	$3, 4$	$4, 1$
m_4	$4, 1$	$2, 2$	$3, 1$	$1, 4$

- Interpretation: entry $(1, 3)$ in the first row and first column indicates that m_1 ranks w_1 first among the women and that w_1 ranks m_1 third among the men.

The deferred acceptance (DA) algorithm: example

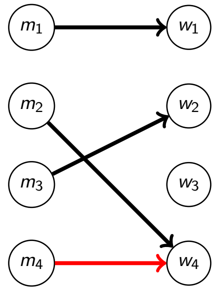
	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



w_1 is the only person to receive multiple proposals; she compares m_1 (rank 3) with m_2 (rank 4) and rejects m_2. Strike this entry from the matrix and iterate.

The deferred acceptance (DA) algorithm: example

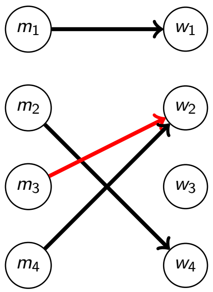
	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



w_4 is the only person to receive multiple proposals; she compares m_2 (rank 2) with m_4 (rank 4) and rejects m_4. Strike this entry from the matrix and iterate.

The deferred acceptance (DA) algorithm: example

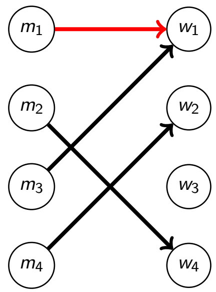
	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



w_2 is the only person to receive multiple proposals; she compares m_3 (rank 4) with m_4 (rank 2) and rejects m_3. Strike this entry from the matrix and iterate.

The deferred acceptance (DA) algorithm: example

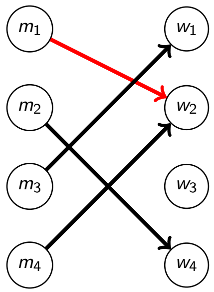
	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



w_1 is the only person to receive multiple proposals; she compares m_1 (rank 3) with m_3 (rank 2) and rejects m_1. Strike this entry from the matrix and iterate.

The deferred acceptance (DA) algorithm: example

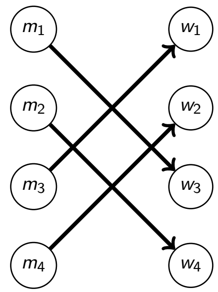
	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



w_2 is the only person to receive multiple proposals; she compares m_1 (rank 3) with m_4 (rank 2) and rejects m_1. Strike this entry from the matrix and iterate.

The deferred acceptance (DA) algorithm: example

	w_1	w_2	w_3	w_4
m_1	1, 3	2, 3	3, 2	4, 3
m_2	1, 4	4, 1	3, 3	2, 2
m_3	2, 2	1, 4	3, 4	4, 1
m_4	4, 1	2, 2	3, 1	1, 4



No rejections; the algorithm stops with stable match

$$(m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_2).$$

The deferred acceptance (DA) algorithm: application

A variant of the marriage problem is the *college admission problem*: each student can be matched to at most one college, but a college can accept many students.
This can be mapped into the marriage problem:

1. Students: one side of the marriage problem, e.g. M.
2. Colleges: other side of the marriage problem, e.g. W. Split college c with quota n into n different women $c_1, \ldots, c_n$.
3. Create artificial preferences by replacing college c in students' rankings by $c_1, \ldots, c_n$, in that order.

The deferred acceptance (DA) algorithm: homework exercise 7

Consider the ranking matrix

	w_1	w_2
m_1	1, 2	2, 1
m_2	2, 1	1, 2

(a) Find a stable matching using the men-proposing DA algorithm.

(b) Find a stable matching using the women-proposing DA algorithm.

(c) Suppose that w_1 lies about her preferences and says that she only finds m_2 acceptable. What is the outcome of the men-proposing DA algorithm now? Verify that both women are better off than under (a): it may pay for the women to lie!