Short solutions precept 2; Mark Voorneveld

Exercise 1:

(a) The only nontrivial information set is $\{(A,C),(A,D)\}$ of player 1. In both nodes of this information set, player 1's experience is $(\emptyset, A, \{(A,C),(A,D)\})$. Since the experience is the same in all nodes of the information set, the game has perfect recall.

(b) Some definitions first:

1. A pure strategy of player 1 is a function that assigns to each information set of player 1 a feasible action. The four pure strategies can be summarized as $S_1 = \{A,B\} \times \{E,F\} = \{(A,E),(A,F),(B,E),(B,F)\}$.
2. A mixed strategy of player 1 is a probability distribution

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)))$$

over these pure strategies: $\sigma_1(s_1) \geq 0$ for all $s_1 \in S_1$ and $\sum_{s_1 \in S_1} \sigma_1(s_1) = 1$.

3. A behavioral strategy of player 1 is a function that assigns to each information set of player 1 a probability distribution over feasible actions. Here, it suffices to specify the probability $p = b_1(\emptyset)(A)$ that 1 assigns to A in the initial node (B has probability $1-p$) and the probability $q = b_1(\{(A,C),(A,D)\})(E)$ that 1 assigns to E in information set $\{(A,C),(A,D)\}$ (F has probability $1-q$).

Let σ_1 be a mixed strategy of player 1. Outcome-equivalent are the behavioral strategies $(p,q) \in [0,1] \times [0,1]$ with $p = \sigma_1((A,E)) + \sigma_1((A,F))$ and $q = \frac{\sigma_1((A,E))}{\sigma_1((A,E))+\sigma_1((A,F))}$ if the denominator is positive and $q \in [0,1]$ arbitrarily otherwise.

(c) Let $(p,q) \in [0,1] \times [0,1]$ be a behavioral strategy of player 1. Outcome-equivalent is the mixed strategy σ_1 with

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)) = (pq, p(1-q), (1-p)q, (1-p)(1-q)).$$

Exercise 2:

(a)

	C	D
(A,E)	1,2	0,0
(A,F)	0,0	5,2
(B,E)	2,5	2,5
(B,F)	2,5	2,5

(b) $((A,F),D)$, $((B,E),C)$, and $((B,F),C))$.

(c) Consecutively eliminate:

1. (A,E): it is strictly dominated by (B,E) and (B,F);
2. C: it is weakly dominated by D;
3. (B,E) and (B,F): they are strictly dominated by (A,F).

The only pure strategy profile that survives this process is $((A,F),D)$.

(d) The game has two subgames: the entire game and a proper subgame starting at the decision node of player 2. The latter has strategic form

	C	D
E	1,2	0,0
F	0,0	5,2

and three Nash equilibria:

1. A pure-strategy equilibrium (E, C). If this is played in the proper subgame, then 1's payoff from A is 1 and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium is $((B, E), C)$. In behavioral strategies: 1 chooses B and E with probability one; 2 chooses C with probability one.
2. A pure-strategy equilibrium (F, D). If this is played in the proper subgame, then 1's payoff from A is 5 and from B is 2, so it is optimal to choose A. Conclude: one subgame perfect equilibrium is $((A, F), D)$. In behavioral strategies: 1 chooses A and F with probability one; 2 chooses D with probability one.
3. A mixed-strategy equilibrium where 1 chooses E with probability 1/2 and 2 chooses C with probability 5/6. If this is played in the proper subgame, then 1's payoff from A is $\frac{5}{6}$ and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium in behavioral strategies is: 1 chooses B with probability 1 and E with probability $\frac{1}{2}$; 2 chooses C with probability $\frac{5}{6}$.

Exercise 3: Summarize an assessment (b, β) by a 4-tuple $(p, q, \alpha_1, \alpha_2) \in [0,1]^4$, where

- p is the probability that 1 chooses In,
- q is the probability that 2 chooses In,
- α_1 is the probability that the belief system assigns to the left node in 1's info set,
- α_2 is the probability that the belief system assigns to the left node in 2's info set.

(a) Distinguish two cases:

1. If $p \in (0,1]$, 2's information set is reached with positive probability. In that case, Bayes' Law dictates that $\alpha_1 = \alpha_2 = \frac{1}{2}$. Conclude: all $(p, q, \alpha_1, \alpha_2) \in (0,1] \times [0,1] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\}$ are Bayesian consistent.
2. If $p = 0$, 2's information set is reached with zero probability and 2 is allowed any belief $\alpha_2 \in [0,1]$ over the nodes in the information set. Bayes' Law only dictates that $\alpha_1 = \frac{1}{2}$. Conclude: all $(p, q, \alpha_1, \alpha_2) \in \{0\} \times [0,1] \times \left\{\frac{1}{2}\right\} \times [0,1]$ are Bayesian consistent.

(b) Every completely mixed profile of behavioral strategies leads to $\alpha_1 = \alpha_2 = \frac{1}{2}$. Conclude: consistent are all $(p, q, \alpha_1, \alpha_2) \in [0,1] \times [0,1] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\}$.

Exercise 4: Denote an assessment by $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha)$. Here, $b_1 = (p_A, p_E) \in [0,1] \times [0,1]$ is 1's behavioral strategy specifying probabilities of choosing A and E in the relevant information sets; $b_2 = p_C \in [0,1]$ is 2's behavioral strategy specifying the probability of choosing C in his information set; belief system β is summarized by the probability $\alpha \in [0,1]$ it assigns to the left node (A, C) in 1's information set $\{(A, C), (A, D)\}$.

Recall: if (b_1, b_2, β) is a sequential equilibrium, (b_1, b_2) is subgame perfect. Using exercise 2(d), we find three sequential equilibria:

1. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1), 1, 1)$;
2. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((1,0), 0, 0)$;
3. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1/2), 5/6, 5/6)$.

Check that these really are limits of completely mixed and Bayesian consistent assessments!

Exercise 5:

(a) Player 1 has pure strategy set $\{(L,L),(L,R),(R,L),(R,R)\}$, where the first letter indicates the action after signal t and the second letter the action after signal t'. Player 2 has pure strategy set $\{(u,u),(u,d),(d,u),(d,d)\}$, where the first letter indicates the action in the left information set (i.e., after player 1 chooses L) and the second letter the action in the right information set. The corresponding strategic form game is

	(u,u)	(u,d)	(d,u)	(d,d)
(L,L)	$\frac{21}{10}, \frac{9}{10}$	$\frac{21}{10}, \frac{9}{10}$	$\frac{1}{10}, \frac{1}{10}$	$\frac{1}{10}, \frac{1}{10}$
(L,R)	$2, \frac{9}{10}$	$\frac{18}{10}, 1$	$\frac{2}{10}, 0$	$0, \frac{1}{10}$
(R,L)	$3, \frac{9}{10}$	$\frac{12}{10}, 0$	$\frac{28}{10}, 1$	$1, \frac{1}{10}$
(R,R)	$\frac{29}{10}, \frac{9}{10}$	$\frac{9}{10}, \frac{1}{10}$	$\frac{29}{10}, \frac{9}{10}$	$\frac{9}{10}, \frac{1}{10}$

There are two pure-strategy Nash equilibria: $((L,L),(u,d))$ and $((R,R),(d,u))$.

(b) The equilibria in (a) are two candidates; but what restrictions do we need on the belief system? There are two nontrivial information sets; the belief system can be summarized by probability α_1 assigned to the top node in the left information set of player 2 and probability α_2 assigned to the top node in the right information set of player 2. Now consider the two candidate pooling equilibria separately:

1. In Nash equilibrium $((L,L),(u,d))$, Bayesian consistency requires $\alpha_1 = \frac{9}{10}$. The right information set is reached with probability zero, so beliefs there are not restricted by Bayesian consistency. But the beliefs do have to be such that player 2 chooses a best response in that information set. The expected payoffs to actions u and d, given the belief α_2, are

$$1 \cdot \alpha_2 + 0 \cdot (1 - \alpha_2) = \alpha_2 \quad \text{and} \quad 0 \cdot \alpha_2 + 1 \cdot (1 - \alpha_2) = 1 - \alpha_2,$$

respectively. Action d is a best response provided $0 \le \alpha_2 \le \frac{1}{2}$.
Conclude: assessments (s_1, s_2, β) with $(s_1, s_2) = ((L,L),(u,d))$ and belief system $\beta = (\alpha_1, \alpha_2) \in \left\{\frac{9}{10}\right\} \times \left[0, \frac{1}{2}\right]$ are pooling equilibria.

2. Similarly, assessments (s_1, s_2, β) with $(s_1, s_2) = ((R,R),(d,u))$ and belief system $\beta = (\alpha_1, \alpha_2) \in \left[\frac{1}{2}, 1\right] \times \frac{9}{10}$ are pooling equilibria.

(c) None, see (a).

Exercise 6: Iteration 1 yields cycle $(2, h_5, 5, h_2, 2)$; iteration 2 yields cycles $(1, h_1, 1)$ and $(4, h_3, 3, h_4, 4)$. The resulting match is $(1, h_1), (2, h_5), (3, h_4), (4, h_3), (5, h_2)$.

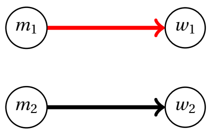
Exercise 7:

(a) One iteration ends with stable match $(m_1, w_1), (m_2, w_2)$.

(b) One iteration ends with stable match $(m_1, w_2), (m_2, w_1)$.

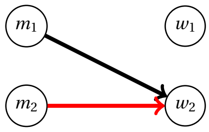
(c) Letting ✓ denote an unacceptable candidate, the ranking matrix with w_1's stated preference is

$$\begin{array}{c|cc|} & w_1 & w_2 \\ \hline m_1 & 1,\checkmark & 2,1 \\ m_2 & 2,1 & 1,2 \\ \hline \end{array}$$ with proposals:



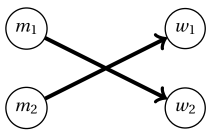
Since w_1 says that m_1 is unacceptable, she rejects him. Strike the entry from the matrix and iterate:

$$\begin{array}{c|cc|} & w_1 & w_2 \\ \hline m_1 & 1,\checkmark & 2,1 \\ m_2 & 2,1 & 1,2 \\ \hline \end{array}$$ with proposals:



Since w_2 receives multiple proposals, she compares m_1 (rank 1) with m_2 (rank 2) and rejects m_2. Strike this entry from the matrix and iterate:

$$\begin{array}{c|cc|} & w_1 & w_2 \\ \hline m_1 & 1,\checkmark & 2,1 \\ m_2 & 2,1 & 1,2 \\ \hline \end{array}$$ with proposals:



Both offers are acceptable; no rejections. The algorithm stops with match $(m_1, w_2), (m_2, w_1)$. Looking at the original preferences, we see that both women get the man they like most. Under (a), they got the man they liked least.